Vertical and Horizontal Asymptotes

**Definition 2.1** The line $x = a$ is a **vertical asymptote** of the function $y = f(x)$ if $y$ approaches $\pm \infty$ as $x$ approaches $a$ from the right or left.

This graph has a vertical asymptote at $x = 1$.

**Definition 2.2** The line $y = b$ is a **horizontal asymptote** of the function $y = f(x)$ if $y$ approaches $b$ as $x$ approaches $\pm \infty$.

This graph has a horizontal asymptote at $x = 1$.

**Example:** Let $f(x) = \frac{1}{x}$. 
The domain of \( f(x) \) is \( D_f = \{ x \in \mathbb{R} \mid x \neq 0 \} \).

Let’s look at how \( f \) behaves near 0.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>−0.1</td>
<td>−10</td>
</tr>
<tr>
<td>−0.01</td>
<td>−100</td>
</tr>
<tr>
<td>−0.001</td>
<td>−1000</td>
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<td>1000</td>
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As the \( x \) values get closer and closer to 0 from the negative side, \( f \)'s values get closer and closer to \(-\infty\). On the other hand, as \( x \) values get closer and closer to 0 from the positive side, \( f \)'s values get larger and larger. This means we have a vertical asymptote at \( x = 0 \).

Now let us look at what \( f \) does as \( x \) gets very large in both directions.

<table>
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<tbody>
<tr>
<td>−10</td>
<td>−0.1</td>
</tr>
<tr>
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<td>−0.01</td>
</tr>
<tr>
<td>−1000</td>
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<td>1000</td>
<td>0.001</td>
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</tbody>
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So as \( x \) goes to either \(+\infty\) or \(−\infty\), the values of \( f \) approach 0. This means we have a horizontal asymptote at \( x = 0 \).

To sketch the rest we can do a table of values. What we end up with is

![Graph](image)

This graph never crosses either of the axes but gets close to both of them.
Transformations of \( \frac{1}{x} \)

Any rational function of the form \( r(x) = \frac{ax + b}{cx + d} \) with \( a, b, c, d \in \mathbb{R} \) can be graphed by shifting, stretching, and/or reflecting the graph of \( f(x) = \frac{1}{x} \). The way we do this is by polynomial division.

**Examples:** Let \( f(x) = \frac{1}{x} \)

1. Sketch \( g(x) = \frac{2}{x - 3} \).

   **solution:** \( g(x) = \frac{2}{x - 3} = 2 \left( \frac{1}{x - 3} \right) = 2f(x - 3) \).

   According to our rules about transforming functions, we can obtain the graph of \( g \) by shifting the graph of \( f(x) = \frac{1}{x} \) by 3 units to the right, and stretching vertically by a factor of 2. Here, \( g(x) \) has a vertical asymptote at \( x = 3 \) and a horizontal asymptote at \( y = 0 \).

![Graph of \( g(x) \)](image)

2. Sketch \( h(x) = \frac{3x + 5}{x + 2} \)

   **solution:** Here we start by dividing the two polynomials

   
   \[
   \begin{array}{c|cc}
   & 3 & \\
   \hline
   x + 2) & 3x + 5 & \\
   & -3x - 6 & \\
   \hline
   & -1 & \\
   \end{array}
   \]

   The remainder is \(-1\), so

   \[
   \frac{3x + 5}{x + 2} = 3 + \frac{-1}{x + 2} = -f(x + 2) + 3
   \]
According to our rules about transforming functions, we can obtain the graph of \( h \) by shifting the graph of \( f(x) = \frac{1}{x} \) by 2 units to the left, shifting 3 units up, and reflecting in the \( x \)-axis. Here, \( h(x) \) has a vertical asymptote at \( x = -2 \) and a horizontal asymptote at \( y = 3 \).

3. Sketch \( r(x) = \frac{2x^2 - 4x + 5}{x^2 - 2x + 1} \).

**solution:** We have to find a few things first.

- **Domain** – the denominator factors into \((x - 1)^2\), so the only \( x \) value that is not allowed is \( x = 1 \). Thus
  \[ D_r = \{ x \in \mathbb{R} \mid x \neq 1 \} \]

- **Vertical asymptotes** – these occur when the denominator is 0. Thus we have a vertical asymptote at \( x = 1 \).

- **Horizontal asymptote** – to find this we divide each term by the highest exponent in the denominator and look at when \( x \to \infty \).
  \[
  r(x) = \frac{2x^2 - 4x + 5}{x^2 - 2x + 1} = \frac{2x - 4 + \frac{5}{x^2}}{x - 2 + \frac{1}{x^2}} = 2 - \frac{4}{x} + \frac{5}{x^2}
  \]
  As \( x \to \infty \), all the terms in that quotient disappear except for the 2 on the top and the 1 on the bottom. Hence, as \( x \to \infty \), \( r(x) \to \frac{2}{1} = 2 \).

- **Behaviour near asymptotes** – now we have to look at what is happening to our function near our asymptotes.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>6.5</td>
<td>14</td>
</tr>
<tr>
<td>0.9</td>
<td>302</td>
</tr>
<tr>
<td>0.99</td>
<td>50,002</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>1.5</td>
<td>14</td>
</tr>
<tr>
<td>1.1</td>
<td>302</td>
</tr>
<tr>
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In general, let \( r(x) \) be a rational function

1. The vertical asymptotes of \( r(x) \) are the roots of the denominator.

2. The horizontal asymptotes are determined as follows:
   - If the degree of the top is larger than the degree of the bottom, there are no horizontal asymptotes.
   - If the degree of the top is smaller than the degree of the bottom, there is a horizontal asymptote at \( y = 0 \).
   - If the degrees of the top and bottom are the same, then there is a horizontal asymptote at \( y = \frac{a}{b} \), where \( a \) is the leading coefficient of the top and \( b \) is the leading coefficient of the bottom.

Example: \( \frac{3x^2 - 2x - 1}{2x^2 + 3x - 2} \)

- Vertical asymptotes – we use the quadratic equation to find the roots of the denominator
  \[
x = \frac{-3 \pm \sqrt{9 - 4(2)(-2)}}{2(2)} = \frac{-3 \pm 5}{4} = -2, \frac{1}{2}
\]
  So \( x = -2 \) and \( x = \frac{1}{2} \) are the vertical asymptotes.

- Horizontal asymptotes – the degrees are the same, so the horizontal asymptote occurs at the ratio of the leading coefficients, in this case the leading term on the top is \( 3x^2 \) while that on the bottom is \( 2x^2 \), so the ratio of the coefficients is \( \frac{3}{2} \). Thus the horizontal asymptote is at \( y = \frac{3}{2} \).
Sketching Graphs of Rational Functions

- Find the domain of $f$ by factoring the denominator.
- Factor the numerator.
- Find the $x$- and $y$-intercepts.
- Find the vertical asymptotes.
- Find the horizontal asymptotes.
- Analyze the behaviour of $f$ around the asymptotes.
- Sketch the graph.

Example: Sketch $r(x) = \frac{2x^2 + 7x - 4}{x^2 + x - 2}$

- $x^2 + x - 2 = (x - 1)(x + 2)$. So $D_r = \{x \in \mathbb{R} \mid x \neq 1, -2\}$

- 
  \[
  x = \frac{-7 \pm \sqrt{49 - 4(2)(-4)}}{2(2)} = \frac{-7 \pm 9}{4} = -4, \frac{1}{2}
  \]

  This means that $2x^2 + 7x - 4 = 2(x - 1)(x + 4) = (2x - 1)(x + 4)$. Thus so far we have

  \[
  r(x) = \frac{(2x - 1)(x + 4)}{(x - 1)(x + 2)}
  \]

- The $x$-intercepts are the roots of the numerator. In this case, the roots are $x = -4$ and $x = \frac{1}{2}$.

  The $y$-intercept we get by subbing in $x = 0$:

  \[
  r(x) = \frac{2(0)^2 + 7(0) - 4}{0^2 + 0 - 2} = 2
  \]

  So the $y$-intercept is $y = 2$.

- We have vertical asymptotes where the denominator is 0, ie, at $x = 1$ and $x = -2$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-2$ from the left</th>
<th>$-2$ from the right</th>
<th>1 from the left</th>
<th>1 from the right</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r(x)$</td>
<td>$(-)(+)$</td>
<td>$(-)(+)$</td>
<td>$(+)(+)$</td>
<td>$(+)(+)$</td>
</tr>
<tr>
<td>$y$</td>
<td>$-\infty$</td>
<td>$\infty$</td>
<td>$-\infty$</td>
<td>$\infty$</td>
</tr>
</tbody>
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