Self-Similar Graph Actions and Partial Crossed Products

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Self-Similar Actions

G group, X finite set, X^* words in X (including an empty word)

Suppose we have an action of G on X^* and a restriction $G \times X \to G$

$$(g, x) \mapsto g|_x .$$

such that the action on X^* can be defined recursively

$$g(x\alpha) = (gx)(g|_x \alpha)$$

The pair (G, X) is called a self-similar action.
Restriction extends to words

$$g|_{\alpha_1\alpha_2\cdots\alpha_n} := g|_{\alpha_1}|_{\alpha_2}\cdots|_{\alpha_n}$$

$$g(\alpha\beta) = (g\alpha)(g|_\alpha \beta)$$

Example: The Odometer

$G = \mathbb{Z} = \langle z \rangle$
$X = \{0, 1\}$

Then the action of $\mathbb{Z}$ on X^* is determined by

$$z0 = 1 \qquad z|_0 = e$$

$$z1 = 0 \qquad z|_1 = z$$

A word α in X^* corresponds to an integer in binary (written backwards), and z adds 1 to α, ignoring carryover.

$$z(001) = 101 \qquad z|_{001} = e$$

$$z^2(011) = 000 \qquad z^2|_{011} = z$$

Self-Similar Actions

(G, X) – self-similar action
Σ_X – infinite words in X.

The action of G on X^* induces an action on Σ_X:
If $\alpha \in \Sigma_X$, then

$$(g\alpha)_n = g|_{\alpha_1 \cdots \alpha_{n-1}} \alpha_n$$

Each $g \in G$ is a homeomorphism on Σ_X (product topology).

Odometer: $\mathbb{Z}$ acts by the usual odometer transformation $\lambda : \Sigma_X \to \Sigma_X$

Self-Similar Actions

(G, X) self-similar action

$\mathcal{T}(G, X)$ is the universal C*-algebra generated by elements

$$\{u_g\}_{g\in G}, \qquad \{s_x\}_{x\in X}, \qquad \text{such that}$$

1. u_g is unitary for all $g \in G$ $(u_g u_g^* = 1 = u_g^* u_g)$
2. s_x is an isometry for all $x \in X$ $(s_x^* s_x = 1)$
3. $s_x^* s_y = 0$ if $x \neq y$
4. $u_g s_x = s_{gx} u_{g|_x}$ for all $g \in G, x \in X$.

If α is a word, $s_\alpha := s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{|\alpha|}}$, then

$$\mathcal{T}(G, X) = \overline{\text{span}\{s_\alpha u_g s_\beta^* \mid \alpha, \beta \in X^*, g \in G\}}$$

If we add the condition $\sum s_x s_x^* = 1$, we get the quotient $\mathcal{O}(G, X)$.

Self-Similar Actions

Question: can $\mathcal{T}(G, X)$ and $\mathcal{O}(G, X)$ be written as partial crossed products?

Answer: sometimes!

$\mathcal{T}(G, X)$ is generated by

$$\mathcal{S}(G, X) := \{s_\alpha u_g s_\beta^* \mid \alpha, \beta \in X^*, g \in G\} \cup \{0\}$$

Consists of partial isometries, closed under multiplication, and so forms an inverse semigroup.

$\mathcal{T}(G, X)$ is universal for representations of $\mathcal{S}(G, X)$, and $\mathcal{O}(G, X)$ is universal for tight representations.

Milan, Steinberg (2011) – when the inverse semigroup is strongly E^*-unitary, the answer is yes.

Inverse Semigroups

A semigroup S is called an inverse semigroup if for every element $s \in S$ there is a unique element s^* such that

$$ss^*s = s \qquad \text{and} \qquad s^*ss^* = s^*$$

$E(S)$ = set of idempotents, that is, elements e such that $e^2 = e$.

It is true that

- Idempotents are self-inverse ($e^* = e$)
- If $e, f \in E(S)$, then $ef \in E(S)$ and $ef = fe$
- For every $s \in S$, we have $s^*s, ss^* \in E(S)$
- $(s^*)^* = s$
- $(st)^* = t^*s^*$

Strongly E^*-unitary Inverse Semigroups

S – inverse semigroup with zero
G – group
A function $\phi : S \setminus \{0\} \to G$ is called a prehomomorphism if

$$\phi(st) = \phi(s)\phi(t) \qquad \text{whenever } st \neq 0$$

$U(S)$ – group generated by the set S subject to the relations $s \cdot t = st$ whenever $st \neq 0$. This is the universal group of S

$\sigma : S \setminus \{0\} \to U(S)$
$\sigma(s) = s$ is a prehomomorphism.

Strongly E^*-unitary Inverse Semigroups

$\phi : S \setminus \{0\} \to G$ prehomomorphism

If $e^2 = e$, then $\phi(e) = 1_G$.

ϕ is idempotent pure if $\phi^{-1}(1_G) = E(S) \setminus \{0\}$.

Definition

An inverse semigroup with zero S is called strongly E^*-unitary if there exists a group G and an idempotent pure prehomomorphism $\phi : S \setminus \{0\} \to G$.

This is equivalent to saying $\sigma : S \setminus \{0\} \to U(S)$ is idempotent pure

Partial Crossed Products

S – strongly E^*-unitary inverse semigroup with zero

$\widehat{E}_0(S)$ – spectrum of S
$\widehat{E}_{\text{tight}}(S)$ – tight spectrum of S

Milan, Steinberg (2011) – exist partial actions of $U(S)$ on $\widehat{E}_0(S)$ and $\widehat{E}_{\text{tight}}(S)$ such that

$$C^*_u(S) \cong C_0(\widehat{E}_0(S)) \rtimes U(S)$$

$$C^*_{\text{tight}}(S) \cong C_0(\widehat{E}_{\text{tight}}(S)) \rtimes U(S)$$

Self-Similar Actions

$$\mathcal{S}(G,X) = \{s_\alpha u_g s_\beta^* \mid \alpha, \beta \in X^*, g \in G\} \cup \{0\}$$

$$\mathcal{T}(G,X) \cong C_u^*(\mathcal{S}(G,X))$$
$$\mathcal{O}(G,X) \cong C_{\text{tight}}^*(\mathcal{S}(G,X))$$

$$\widehat{E}_0(S) \cong \Sigma_X \cup X^*$$
$$\widehat{E}_{\text{tight}}(S) \cong \Sigma_X$$

Self-Similar Actions

(G, X) is called residually free if whenever $g \in G$ and $\alpha \in X^*$, then

$$\begin{matrix} g\alpha = \alpha \\ g|_\alpha = 1_G \end{matrix} \implies g = 1_G$$

Proposition

$\mathcal{S}(G, X)$ strongly E^*-unitary $\iff$ (G, X) residually free

Corollary

(G, X) residually free $\implies$ $\mathcal{T}(G, X), \mathcal{O}(G, X)$ are partial crossed products

Example: The Odometer

$(\mathbb{Z}, \{0,1\})$ – The Odometer

If $z^n\alpha = \alpha$, then n is a multiple of $2^{|\alpha|}$.
If $z^n|_\alpha = e$, then $|n| < 2^{|\alpha|}$

$\Rightarrow (\mathbb{Z}, \{0,1\})$ is residually free.

If we write $H := U(\mathcal{S}(\mathbb{Z}, \{0,1\}))$

Then, $\mathcal{O}(G, X) \cong C(\Sigma_{\{0,1\}}) \rtimes H$.

What is H?

What is the action?

Example: The Odometer

$$\sigma : \mathcal{S}(\mathbb{Z}, \{0,1\}) \setminus \{0\} \to H$$

The images of s_0, s_1 and z generate H

$$\sigma(s_0) := a, \quad \sigma(s_1) := b, \quad \sigma(z) := Z$$

$$Za = b, \quad Zb = aZ$$

$$Z = ba^{-1}, \quad Z = aZb^{-1}$$

$$H = \left\langle a, b \mid ba^{-1} = aba^{-1}b^{-1} \right\rangle$$

$$H = \left\langle a, b \mid ba^{-1} = a^n ba^{-1} b^{-n} \text{ for all } n \in \mathbb{Z} \right\rangle$$

One can show that elements of H of the form $\alpha\beta^{-1}$ with $|\alpha| = |\beta|$ are images of powers of z.

Example: The Odometer

$$H = \left\langle a, b \mid ba^{-1} = a^n ba^{-1} b^{-n} \text{ for all } n \in \mathbb{Z} \right\rangle$$

Descrption of the partial homeomorphisms $\{\theta_g\}_{g \in H}$ on Σ_X:

If $\alpha \in \{a, b\}^*$, let $\tilde{\alpha} \in \{0, 1\}^*$ with $a \to 0$, and $b \to 1$.

$$\theta_\alpha : \Sigma_X \to \tilde{\alpha}\Sigma_X$$

$$\theta_\alpha(y) = \tilde{\alpha} y$$

$$\theta_{\beta^{-1}} : \tilde{\beta}\Sigma_X \to \Sigma_X$$

$$\theta_{\beta^{-1}}(\tilde{\beta} y) = y$$

If $|\alpha| = |\beta|$, then

$$\theta_{\alpha\beta^{-1}} = \lambda^{n_{\tilde{\alpha}} - n_{\tilde{\beta}}}$$

where $n_{\tilde{\alpha}}$ is the integer equal to $\tilde{\alpha}$ in binary (backwards).

Self-Similar Graph Actions

Exel, Pardo (2013) – generalized the construction of self-similar actions to finite paths in a graph.

$E = (E^0, E^1, r, d)$ – finite graph
E^* – finite paths in E (including vertices)

A self-similar action of a group G on E is an action of graph automorphisms which extends to E^* recursively:

$$g(e\alpha) = ge(g|_e \alpha)$$

$$\mathcal{S}(G, E) = \{s_\alpha u_g s_\beta^* \mid \alpha, \beta \in E^*, g \in G, d(\alpha) = gd(\beta)\}$$

Proposition

$\mathcal{S}(G, E)$ *strongly* E^**-unitary* $\iff$ (G, E) *residually free*