University of Ottawa

Department of Mathematics and Statistics

MAT 1302E : Mathematical Methods II

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Second Midterm Exam – Version B

November 21, 2013

Surname ____________________ First Name ____________________

Student # ____________________ DGD (Monday/Thursday) ____________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(d) Write your student number at the top of each page in the space provided.
(e) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(f) You are strongly recommended to write in **pen**, not pencil.
(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	5	3	3	3	4	5	23
Grade							

1. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 1 & 3 & -2 \\ -2 & 0 & 1 & -3 \\ 0 & 2 & 4 & 5 \end{bmatrix}.$$

(a) **[3 points]** Calculate $\det A$ by using the cofactor expansion method.

Solution: We use cofactor expansion along the second column:

$$\det A = -0\begin{vmatrix} -1 & 3 & -2 \\ -2 & 1 & -3 \\ 0 & 4 & 5 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 & 2 \\ -2 & 1 & -3 \\ 0 & 4 & 5 \end{vmatrix} - 0\begin{vmatrix} 1 & -1 & 2 \\ -1 & 3 & -2 \\ 0 & 4 & 5 \end{vmatrix} + 2\begin{vmatrix} 1 & -1 & 2 \\ -1 & 3 & -2 \\ -2 & 1 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 & 2 \\ -2 & 1 & -3 \\ 0 & 4 & 5 \end{vmatrix} + 2\begin{vmatrix} 1 & -1 & 2 \\ -1 & 3 & -2 \\ -2 & 1 & -3 \end{vmatrix}.$$

Now we have

$$\begin{vmatrix} 1 & -1 & 2 \\ -2 & 1 & -3 \\ 0 & 4 & 5 \end{vmatrix} = 0\begin{vmatrix} -1 & 2 \\ 1 & -3 \end{vmatrix} - 4\begin{vmatrix} 1 & 2 \\ -2 & -3 \end{vmatrix} + 5\begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} = (-4)(1) + (5)(-1) = -9 \text{ (last row expansion)}$$

and

$$\begin{vmatrix} 1 & -1 & 2 \\ -1 & 3 & -2 \\ -2 & 1 & -3 \end{vmatrix} = 1\begin{vmatrix} 3 & -2 \\ 1 & -3 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -2 \\ -2 & -3 \end{vmatrix} + 2\begin{vmatrix} -1 & 3 \\ -2 & 1 \end{vmatrix} = (-1)(-7) + (1)(-1) + (2)(5) = 2. \text{ (first row)}$$

Therefore $\det A = (-9) + 4 = -5$.

(b) [**2 points**] This part is independent of part (a).

Let P, Q, R be 3×3 matrices such that $\det(P) = -\frac{1}{2}$, $\det(Q) = 5$, and $\det(R) = -10$. Compute $\det(-\frac{1}{2}P^2RQ^{-2}R^T)$.

Solution:

$$\det(-\frac{1}{2}P^2RQ^{-2}R^T) = (-\frac{1}{2})^3(-\frac{1}{2})^2(-10)(\frac{1}{5})^2(-10) = -\frac{1}{8}.$$

2. [**3 points**] Let $A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 0 & 1 \end{bmatrix}$. Solve the following matrix equation for X:

$$A(X^T - C) = -BC$$

Solution:

$$\begin{aligned} A(X^T - C) = -BC \;&\Rightarrow\; A^{-1}A(X^T - C) = -A^{-1}BC \;\Rightarrow\; X^T - C = -A^{-1}BC \\ &\Rightarrow\; X^T = C - A^{-1}BC \;\Rightarrow\; X = C^T - C^T B^T (A^{-1})^T \end{aligned}$$

Now we have

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \;\Rightarrow\; (A^{-1})^T = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

and therefore

$$C^T B^T (A^{-1})^T = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 8 \\ 0 & 0 \\ 19 & 9 \end{bmatrix}$$

and

$$X = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 18 & 8 \\ 0 & 0 \\ 19 & 9 \end{bmatrix} = \begin{bmatrix} -17 & -6 \\ 0 & 0 \\ -16 & -8 \end{bmatrix}$$

3. Determine if the following sets are subspaces of $\mathbb{R}^3$ or not. Justify your answer.

(a) [**1 point**]

$$U = \left\{ \begin{bmatrix} a+b \\ -b+2c \\ -2c \end{bmatrix} \middle|\, a,\, b,\, c \in \mathbb{R} \right\}$$

Solution: U is a subspace because every vector in V can be written as

$$\begin{bmatrix} a+b \\ -b+2c \\ -2c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$$

and therefore

$$U = \operatorname{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \right\}$$

(b) [**1 point**] $V = \left\{ \begin{bmatrix} a^2+b^2 \\ b-c \\ c \end{bmatrix} \middle|\, a,\, b,\, c \in \mathbb{R} \right\}$

Solution: The top entry of every vector in V is $a^2+b^2 \geq 0$. Therefore V is not a subspace because it is not closed under scaling:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ is in } V \text{ for } a = 1, b = c = 0$$

but

$$(-1) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \text{ is not in } V.$$

(c) [**1 point**]

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; x_1, x_2, x_3 \in \mathbb{R} \text{ and } \begin{bmatrix} 1 & 1 & -3 \\ -2 & -7 & 2 \\ 5 & 15 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Solution: Note that $W = \operatorname{Nul} A$ where A is the given 3×3 matrix. Therefore A is a subspace.

4. [**3 points**] Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & -1 \\ 1 & 0 & -\frac{1}{2} \end{bmatrix}.$$

Solution:

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 1 & 0 & -\frac{1}{2} & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & -2 & \frac{1}{2} & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & -1 & 1 & 1 \end{array}\right] \xrightarrow{R_3 \to -2R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & -2 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 + R_3]{R_1 \to R_1 + R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 3 & -2 & -2 \\ 0 & 2 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & 2 & -2 & -2 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 3 & -2 & -2 \\ 0 & 1 & 0 & 1 & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & 2 & -2 & -2 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & 2 & -2 & -2 \end{array}\right]$$

Therefore $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -1 \\ 2 & -2 & -2 \end{bmatrix}$

5. Let $A = \begin{bmatrix} 1 & -1 & 2 & 0 & 3 \\ 0 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 2 & -1 & 1 \end{bmatrix}$.

(a) [**3 points**] Find a basis for Nul A. **Solution:** The reduced echelon form of A is as follows:

$$\begin{bmatrix} 1 & -1 & 2 & 0 & 3 \\ 0 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 2 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} 1 & -1 & 2 & 0 & 3 \\ 0 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 3R_3} \begin{bmatrix} 1 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to -R_2} \begin{bmatrix} 1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

In the corresponding homogeneous linear system, the variables x_1, x_3, and x_5 are basic, and x_2, x_4 are free. The general solution is

$$\begin{cases} x_1 = x_2 - x_4 \\ x_3 = \frac{1}{2} x_4 \\ x_5 = 0 \end{cases}$$

and its vector parametric form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}.$$

Therefore a basis for Nul A is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(b) [**1 point**] What is rank A? Justify your answer.

Solution: By the rank-nullity theorem we know that

$$\operatorname{rank} A + \dim \operatorname{Nul} A = 5 \ \Rightarrow \ \operatorname{rank} A = 5 - 2 = 3.$$

6. An economy consists of two sectors: the Construction sector and the Service sector. We know that:

- For each unit of output, Construction requires $\frac{1}{3}$ units from itself and $\frac{1}{2}$ units from Service.
- For each unit of output, Service requires $\frac{1}{6}$ units from Construction and $\frac{3}{4}$ units from iteself.

(a) [**1 point**] What is the consumption matrix C for this economy?

(b) [**2 points**] Find $I - C$ and $(I - C)^{-1}$.

(c) [**1 point**] Determine what intermediate demands are created if Construction wants to produce 6 units and Service wants to produce 36 units.

(d) [**1 point**] Find the production levels that will satisfy the final demand of 3 units from Construction and 12 units from Service.

Solution: (a) The consumption matrix of this economy is:

$$C = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}.$$

(b)

$$I - C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix}.$$

$$(I - C)^{-1} = \frac{1}{\frac{2}{12} - \frac{1}{12}} \begin{bmatrix} \frac{1}{4} & \frac{1}{6} \\ \frac{1}{2} & \frac{2}{3} \end{bmatrix} = 12 \begin{bmatrix} \frac{1}{4} & \frac{1}{6} \\ \frac{1}{2} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 6 & 8 \end{bmatrix}.$$

(c)

$$6 \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \end{bmatrix} + 36 \begin{bmatrix} \frac{1}{6} \\ \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 8 \\ 30 \end{bmatrix}$$

(d) Let x_1 and x_2 be the outputs of Construction and Service respectively. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the production vector. Note that $\mathbf{d} = \begin{bmatrix} 3 \\ 12 \end{bmatrix}$.

$$\mathbf{x} = (\mathbf{I} - \mathbf{C})^{-\mathbf{1}}\mathbf{d} = \begin{bmatrix} 3 & 2 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 12 \end{bmatrix} = \begin{bmatrix} 33 \\ 114 \end{bmatrix}.$$