MAT 1302 E, Fall 2013

Homework 4

Professor: Catalin Rada

DUE DATE December 3, 2013, by 7:00p.m. in the drop box in 585 King Edward ave. (KED)

For all of the questions below, you must show each step in any row reduction and state what operation (if any) you are performing at each step.

1. **(2 points)** Consider the following complex numbers $z = -2+2i$ et $w = 1+3i$. Express the complex number z/w in the form $a + bi$.

Solution: One has that:

$$\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}} = \frac{z\overline{w}}{|w|^2}$$

$$= \frac{(-2+2i)(1-3i)}{1^2+3^2} = \frac{-2+6i+2i-6i^2}{10} = \frac{4+8i}{10} = \frac{2}{5} + \frac{4}{5}i.$$

2. Consider the following matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

(a) **(1.5 point)** Find the characteristic polynomial of A.
(b) **(2 points)** Find the eigenvalues of A and their multiplicities.
(c) **(3 points)** Find a basis for every eigenspace associated to the eigenvalues found in (b).
(d) **(1.5 points)** Is A diagonalizable? If it is, justify your answer and find an invertible matrix P and a diagonal matrix D such that $A = P^{-1}DP$.

Solution (a) $P_A(\lambda) = \det(A - \lambda I_3) = \det \begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix}$. Expanding across column 3 one has that:

$P_A(\lambda) = (-1)^{3+3}(1-\lambda)\det \begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix}$, hence $P_A(\lambda) = (1-\lambda)(\lambda^2 - 2\lambda) = -\lambda(1-\lambda)(2-\lambda)$.

(b) $P_A(\lambda) = -\lambda(1-\lambda)(2-\lambda) = 0$ if and only if $\lambda_1 = 0$, $\lambda_2 = 1$ or $\lambda_3 = 2$, which are the eigenvalues of A. Multiplicity of each eigenvalue is 1.

(c)

- For $\lambda_1 = 0$ we have

$$A - \lambda_1 I_3 = A \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{L_2 \leftrightarrow L_3} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 + R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So x_3 is free in $Ax = 0$ and we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

Thus the eigenspace is $span\{\begin{bmatrix}-1\\-1\\1\end{bmatrix}\}$ with basis $\left\{\begin{bmatrix}-1\\-1\\1\end{bmatrix}\right\}$.

- For λ_2 we get

$$A-\lambda_2 I_3 = A - I_3 = \begin{bmatrix}0 & -1 & 1\\-1 & 0 & 0\\1 & 0 & 0\end{bmatrix} \xrightarrow{R_1\leftrightarrow R_2} \begin{bmatrix}-1 & 0 & 0\\0 & -1 & 0\\1 & 0 & 0\end{bmatrix} \xrightarrow{R_3\to R_3+R_1} \begin{bmatrix}-1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 0\end{bmatrix}$$

$$\xrightarrow[R_2\to -R_2]{R_1\to -R_1} \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}.$$

So in $(A-I_3)x=0$, variable x_3 is free, and we have

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\x_3\end{bmatrix} = x_3\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Hence the eigenspace is $span\{\begin{bmatrix}0\\0\\1\end{bmatrix}\}$ with basis $\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$.

- For $\lambda_3 = 2$ we have

$$A-\lambda_3 I_3 = \begin{bmatrix}-1 & -1 & 0\\-1 & -1 & 0\\1 & 0 & -1\end{bmatrix} \xrightarrow[R_3\to R_3+R_1]{R_2\to R_2-R_1} \begin{bmatrix}-1 & -1 & 0\\0 & 0 & 0\\0 & -1 & -1\end{bmatrix} \xrightarrow{R_2\leftrightarrow R_3} \begin{bmatrix}-1 & -1 & 0\\0 & -1 & -1\\0 & 0 & 0\end{bmatrix}$$

$$\xrightarrow{R_2\to -R_2} \begin{bmatrix}-1 & -1 & 0\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix} \xrightarrow{R_1\to R_1+R_2} \begin{bmatrix}-1 & 0 & 1\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix} \xrightarrow{L_1\to -L_1} \begin{bmatrix}1 & 0 & -1\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}.$$

One has that in $(A-2I_3)x=0$ the variable x_3 is free and we have that

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_3\\-x_3\\x_3\end{bmatrix} = x_3\begin{bmatrix}1\\-1\\1\end{bmatrix}.$$

So the eignspace is $\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\}$ with basis $\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$.

(d) Yes, A is diagonalizable since it has 3 distinct eigenvalues. Moreover $P = \begin{bmatrix}-1 & 0 & 1\\-1 & 0 & -1\\1 & 1 & 1\end{bmatrix}$ and

$$D = \begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 2\end{bmatrix}.$$

3. In a group of happy students, we conducted a survey about the impact of doing well at tests among these students. The survey showed that if a student did well in a test, then there is one chance in four that he/she will not do well in the next test. If a student did not do well in a test, then there is one chance in two that he/she will do well in the next test. Daniel is one if these students and always does well in the first test.

(a) **(1 point)** Find the stochastic matrix.
(b) **(1 point)** What are the chances that Daniel will not do well in test 3.
(c) (**2 points**) Find the chances of success and of failure in the long-run.