MAT 1302E, Fall 2013

Assignment 3

Professor: Catalin Rada

Submission deadline: November 12, 2013 at 7:00p.m.

Show your work! For all of the questions below, you must show each step in any row reduction and state what operation you are performing at each step.

Do NOT leave submission of your assignment until the last minute. Late assignments will NOT be accepted. The drop box is found on the first floor of 585 King Edward ave

1. An economy has two sectors: Agriculture and Construction. The sectors satisfy the following conditions:

   - For each unit of output, Agriculture requires .2 units of itself and .5 units from Construction.
   - For each unit of output, Construction requires .6 units from Agriculture.

   (a) **(1 point)** Determine the consumption matrix C for this economy.

   Solution:

$$C = \begin{bmatrix} 0.2 & 0.6 \\ 0.5 & 0 \end{bmatrix}.$$

   (b) **(2 points)** Calculate the inverse of the matrix $(I_2 - C)$.

   Solution:

$$I_2 - C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.6 \\ 0.5 & 0 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.6 \\ -0.5 & 1 \end{bmatrix}.$$

$$(I_2 - C)^{-1} = \frac{1}{(0.8)(1) - (-0.5)(-0.6)} \begin{bmatrix} 1 & 0.6 \\ 0.5 & 0.8 \end{bmatrix} = \frac{1}{0.5} \begin{bmatrix} 1 & 0.6 \\ 0.5 & 0.8 \end{bmatrix} = \begin{bmatrix} 2 & 1.2 \\ 1 & 1.6 \end{bmatrix}.$$

   (c) **(2 points)** Using your answer to part (b), find the production levels necessary to meet the final demand of 100 units from Agriculture and 200 units from Construction.

   Solution: Let x_1 and x_2 denote the output levels of Agriculture and Construction, and let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the production vector. The final demain is $\vec{d} = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$. Therefore we have

$$\vec{x} = (I_2 - C)^{-1}\vec{d} = \begin{bmatrix} 2 & 1.2 \\ 1 & 1.6 \end{bmatrix} \begin{bmatrix} 100 \\ 200 \end{bmatrix} = \begin{bmatrix} 440 \\ 420 \end{bmatrix}.$$

   Thus the production levels to meet the final demand of 100 units from Agriculture and 200 units from Construction are: 440 units from Agriculture and 420 units from Construction.

2. Consider the matrix $A = \begin{bmatrix} 1 & 3 & -1 & 1 \\ -1 & -5 & 5 & 0 \\ 4 & 7 & 6 & 1 \end{bmatrix}$. The reduced echelon form of A is

$$\begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) **[2 points]** Find a basis for $\operatorname{Col} A$. Then determine $\operatorname{rank} A$.

Solution: The pivot columns of A are the 1st, the 2nd, and the 4th. Therefore a basis for $\operatorname{Col} A$ is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

It follows that $\operatorname{rank} A = 3$.

(b) **[2 points]** Find a basis for $\operatorname{Nul} A$. What is the dimension of $\operatorname{Nul} A$?

Solution: To find a basis for $\operatorname{Nul} A$ we need to write the general solution to $A\vec{x} = \vec{0}$ in the vector parametric form. By the reduced echelon form of A we have

$$\begin{cases} x_1 = -5x_3 \\ x_2 = 2x_3 \\ x_3 = \text{free} \\ x_4 = 0 \end{cases}$$

Therefore the vector parametric form of the solution of $A\vec{x} = \vec{0}$ is given by

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5x_3 \\ 2x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad x_3 \in \mathbb{R}.$$

Consequently, a basis for $\operatorname{Nul} A$ is given by $\left\{ \begin{bmatrix} -5 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Moreover, $\dim \operatorname{Nul} A = 1 = \#$ of free variables.

3. Determine if each of the following subsets of $\mathbb{R}^3$ is indeed a subspace of $\mathbb{R}^3$ or not. You should justify your answers.

(a) **[2 points]**

$$U = \left\{ \begin{bmatrix} 2a + c \\ a - 2b \\ a - c \end{bmatrix} \middle| a,\, b,\, c \in \mathbb{R} \right\}$$

Solution: We prove that U is a subspace of $\mathbb{R}^3$ by showing that it is the span of a given list of vectors. We can write $\begin{bmatrix} 2a + c \\ a - 2b \\ a - c \end{bmatrix} \in U$ as

$$\begin{bmatrix} 2a + c \\ a - 2b \\ a - c \end{bmatrix} = \begin{bmatrix} 2a \\ a \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ -2b \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ 0 \\ -c \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

where $a,\, b,\, c \in \mathbb{R}$. Therefore U is the set of all linear combinations of $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Thus $U = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$.

(b) **[2 points]**

$$V = \left\{ \begin{bmatrix} a^2 \\ b-a \\ a \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$$

Solution: V is not a subspace of $\mathbb{R}^3$ for each of the following reasons:

(a) V is not closed under scalar multiplication. For example, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \in V$ $(a = b = 1)$, but

$$-\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \notin V,$$

since the first entry of this vector is not the square of its last entry.

(b) V is not closed under addition. For example, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \in V$, but

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} \notin V,$$

since the first entry of this vector is not the square of its last entry.

4. *This question has two independent parts. You should justify your answer for each part.*

(a) **(2 points)** Assume that $a, b, \ldots, h, i \in \mathbb{R}$ are given such that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -3.$$

Calculate

$$\begin{vmatrix} -a & -d & -g \\ c & f & i \\ 2b-c & 2e-f & 2h-i \end{vmatrix}.$$

Hint: Use the properties of determinant in relation with row operations, etc., to reach from the second determinant to the first determinant.

Solution: We have

$$\begin{vmatrix} -a & -d & -g \\ c & f & i \\ 2b-c & 2e-f & 2h-i \end{vmatrix} \overset{R_3 \to R_3 + R_2}{=} \begin{vmatrix} -a & -d & -g \\ c & f & i \\ 2b & 2e & 2h \end{vmatrix}$$

$$\overset{\substack{R_1 \to -R_1 \\ R_3 \to 1/2 R_3}}{=} -2 \begin{vmatrix} a & d & g \\ c & f & i \\ b & e & h \end{vmatrix} \overset{R_2 \leftrightarrow R_3}{=} -(-2) \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$

$$\overset{\text{Transpose}}{=} -(-2) \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -(-2)(-3) = -6\,.$$

(b) **(2 points)** Let A, B and C be 3×3 matrices with

$$\det A = -2, \quad \det B = -3, \quad \det\big(-2AC^{-1}B^2A^TB^{-1}C^2\big) = 32.$$

Calculate $\det C$.

Solution: We have

$$32 = \det\big(-2AC^{-1}B^2A^TB^{-1}C^2\big) = (-2)^3 \det(A)\det(C^{-1})\det(B^2)\det(A^T)\det(B^{-1})\det(C^2)$$

$$= -8\det(A)\frac{1}{\det(C)}(\det(B))^2\det(A)\frac{1}{\det(B)}(\det(C))^2 = -8(\det(A))^2\det(B)\det(C) = 96\det(C)$$

Therefore $\det C = \frac{32}{96} = 1/3$.