MAT 1302 E, Fall 2013

Assignment 1

Professor: Catalin Rada

Due date: September 24, 2013, by 4:00p.m. in the drop box in 585 King Edward ave. (KED)

For all of the questions below, you must show each step in any row reduction. At each step you should clearly write which operation is being used.

Do NOT leave submission of your assignment until the last minute. Late assignments will NOT be accepted.

1. **(4 points)** Solve the following system using the method of row reduction. Check your answer.

$$\begin{array}{rcrcrcl} -2x_1 & + & x_2 & - & x_3 & = & 5 \\ & & x_2 & - & x_3 & = & 4 \\ x_1 & - & x_2 & - & x_3 & = & \frac{3}{2} \end{array}$$

Solution: We start with the augmented matrix and row reduce it to obtain the *reduced echelon form.*

$$\left[\begin{array}{ccc|c} -2 & 1 & -1 & 5 \\ 0 & 1 & -1 & 4 \\ 1 & -1 & -1 & \frac{3}{2} \end{array}\right] \xrightarrow{R_1 \to -\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{5}{2} \\ 0 & 1 & -1 & 4 \\ 1 & -1 & -1 & \frac{3}{2} \end{array}\right] \xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{5}{2} \\ 0 & 1 & -1 & 4 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & 4 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{5}{2} \\ 0 & 1 & -1 & 4 \\ 0 & 0 & -2 & 6 \end{array}\right] \xrightarrow{R_3 \to -\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{5}{2} \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 - \frac{1}{2}R_3]{R_2 \to R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right] \xrightarrow{R_1 \to R_1 + \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

Switching back to equation notation gives

$$\begin{aligned} x_1 &= -\frac{1}{2} \\ x_2 &= 1 \\ x_3 &= -3 \end{aligned}$$

and so the system has the unique solution $(-\frac{1}{2}, 1, -3)$. To check our answer, we substitute this solution back into the original equations:

$$\begin{array}{ccccccl} (-2)(-\frac{1}{2}) & + & (1)(1) & + & (-1)(-3) & = 5 \\ (0)(-\frac{1}{2}) & + & (1)(1) & + & (-1)(-3) & = 4 \\ (1)(-\frac{1}{2}) & + & (-1)(1) & + & (-1)(-3) & = \frac{3}{2} \end{array}$$

Since each substitution yields a true statement, we have verified that $(-\frac{1}{2}, 1, -3)$ is indeed a solution.

2. **(4 points)** The following matrix is the augmented matrix of a linear system.

$$\left[\begin{array}{cccc|c} 1 & 3 & -1 & -1 & 1 \\ 0 & -2 & 4 & -2 & 0 \\ -1 & -2 & -1 & 2 & -3 \end{array}\right]$$

Determine the pivot columns of the above matrix. Determine if the linear system is consistent or inconsistent. You should justify your answers. (You do not need to completely solve the linear system if it is consistent.)

Solution: We row reduce to obtain an echelon form.

$$\left[\begin{array}{cccc|c} 1 & 3 & -1 & -1 & 1 \\ 0 & -2 & 4 & -2 & 0 \\ -1 & -2 & -1 & 2 & -3 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_1} \left[\begin{array}{cccc|c} 1 & 3 & -1 & -1 & 1 \\ 0 & -2 & 4 & -2 & 0 \\ 0 & 1 & -2 & 1 & -2 \end{array}\right] \xrightarrow{R_3 \to R_3 + \frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 3 & -1 & -1 & 1 \\ 0 & -2 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$$

The pivot columns are the first, second, and fifth columns. Since the rightmost column is a pivot column, the system is inconsistent.

3. **(5 points)** Find the general solution of the following system. Indicate which variables are basic and which are free. Check your final answer.

$$\begin{aligned} x_1 + \frac{1}{3}x_2 &= -x_3 + 4x_4 - 9 \\ 2x_4 &= x_1 + \frac{1}{3}x_2 + 4 \\ x_1 + \frac{1}{3}x_2 + 2x_3 &= 6x_4 - 14 \end{aligned}$$

Solution: The above system is equivalent to the following system:

$$\begin{array}{rrrrl} x_1 & +\frac{1}{3}x_2 & +x_3 & -4x_4 & = -9 \\ -x_1 & -\frac{1}{3}x_2 & & +2x_4 & = 4 \\ x_1 & +\frac{1}{3}x_2 & +2x_3 & -6x_4 & = -14 \end{array}$$

Next we write the the augmented matrix of the latter system and then row reduce.

$$\left[\begin{array}{cccc|c} 1 & \frac{1}{3} & 1 & -4 & -9 \\ -1 & -\frac{1}{3} & 0 & 2 & 4 \\ 1 & \frac{1}{3} & 2 & -6 & -14 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{cccc|c} 1 & \frac{1}{3} & 1 & -4 & -9 \\ 0 & 0 & 1 & -2 & -5 \\ 0 & 0 & 1 & -2 & -5 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & \frac{1}{3} & 1 & -4 & -9 \\ 0 & 0 & 1 & -2 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{cccc|c} 1 & \frac{1}{3} & 0 & -2 & -4 \\ 0 & 0 & 1 & -2 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The variables x_1 and x_3 correspond to pivot columns, hence they are *basic*. The variables x_2 and x_4 are *free*. Solving for the basic variables in terms of the free ones, we obtain the general solution of the system:

$$\begin{aligned} x_1 &= -4 - \frac{1}{3}x_2 + 2x_4 \\ x_2 &= \text{free} \\ x_3 &= -5 + 2x_4 \\ x_4 &= \text{free}. \end{aligned}$$

We check our answer by substituting our expressions for x_1, x_2 into the original equations (leaving in the free variables x_3, x_4).

$$\begin{aligned}
(-4 - \frac{1}{3}x_2 + 2x_4) + \frac{1}{3}x_2 &= -(-5 + 2x_4) + 4x_4 - 9 \\
2x_4 &= (-4 - \frac{1}{3}x_2 + 2x_4) + \frac{1}{3}x_2 + 4 \\
(-4 - \frac{1}{3}x_2 + 2x_4) + \frac{1}{3}x_2 + 2(-5 + 2x_4) &= 6x_4 - 14
\end{aligned}$$

Since each substitution yields a true statement, we have verified our solution.