MAT 1302 E, Fall 2013

Homework 2

Professor: Catalin Rada

DUE DATE October 29, 2013, by 7:00p.m. in the drop box in 585 King Edward ave. (KED)

For all of the questions below, you must show each step in any row reduction and state what operation you are performing at each step.

1. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & k \\ 1 & -2 & k+1 \end{bmatrix}$.

(a) **(3 points)** Find <u>all</u> values of k for which A is invertible.

Solution:

$$A \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & k-2 \\ 1 & -2 & k+1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & k-2 \\ 0 & -1 & k \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & k-2 \\ 0 & 0 & 2(k-1) \end{bmatrix}$$

A is invertible if there is a pivot position in each row. The first 2 rows do have pivot positions. The third one has a pivot position if

$$2(k-1) \neq 0 \Longrightarrow k \neq 1.$$

So, if $k \neq 1$, then the matrix A is invertible.

(b) **(2.5 points)** Find the inverse of A when $k = 0$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & -2 & -3 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & -2 & -3 & 1 & 1 \end{array}\right] \xrightarrow{R_3 \to \frac{1}{-2} R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{-1}{2} & \frac{-1}{2} \end{array}\right] \xrightarrow{R_1 \to R_1 + R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{-1}{2} & \frac{-1}{2} \end{array}\right] \xrightarrow{R_2 \to R_2 + 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{-1}{2} & \frac{-1}{2} \end{array}\right]$$

Hence

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & -1 \\ \frac{3}{2} & \frac{-1}{2} & \frac{-1}{2} \end{bmatrix}$$

(c) **(1.5 points)** Use part (b) to solve the matrix equation $Ax = b$ where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Solution: Since A is invertible, then the equation $Ax = b$ has a unique solution

$$x = A^{-1}b = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & -1 \\ \frac{3}{2} & \frac{-1}{2} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

2. **(4 points)**

Let $A = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 20 \\ 4 & 10 \end{bmatrix}$ be two matrices. Is there a 2×2 matrix X such that $AB(A + 3BX^T)B^{-1}A^{-1} = I_2$? If the answer is affirmative, find X.

Solution:

- Multiply on the left by A^{-1} and on the right by A:
$$B(A + 3BX^T)B^{-1} = A^{-1}I_2A = (A^{-1}I_2)A = A^{-1}A = I_2$$
- Multiply on the left by B^{-1} and on the right by B:
$$A + 3BX^T = B^{-1}I_2B = (B^{-1}I_2)B = B^{-1}B = I_2$$
- Substract A from the above equality:
$$3BX^T = I_2 - A$$
- Divide by 3 the above equality:
$$BX^T = \frac{1}{3}(I_2 - A)$$
- Multiply by B^{-1} on the left:
$$X^T = \frac{1}{3}B^{-1}(I_2 - A)$$
- Transpose and get:
$$(X^T)^T = X = [\frac{1}{3}B^{-1}(I_2 - A)]^T = \frac{1}{3}[B^{-1}(I_2 - A)]^T = \frac{1}{3}(I_2 - A)^T(B^{-1})^T$$
- Replacing the matrices from statement, one gets:
$$\begin{aligned} X &= \frac{1}{3}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}\right)^T \left(\begin{bmatrix} -1 & 20 \\ 4 & 10 \end{bmatrix}^{-1}\right)^T = \frac{1}{3}\left(\begin{bmatrix} 2 & -2 \\ -1 & 0 \end{bmatrix}\right)^T \left(\frac{-1}{90}\begin{bmatrix} 10 & -20 \\ -4 & -1 \end{bmatrix}\right)^T \\ &= \frac{1}{3}\begin{bmatrix} 2 & -2 \\ -1 & 0 \end{bmatrix}^T \frac{-1}{90}\begin{bmatrix} 10 & -20 \\ -4 & -1 \end{bmatrix}^T = \frac{-1}{270}\begin{bmatrix} 2 & -1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 10 & -4 \\ -20 & -1 \end{bmatrix} \\ &= \frac{-1}{270}\begin{bmatrix} 40 & -7 \\ -20 & 8 \end{bmatrix} \end{aligned}$$

3. **(3 points)** Let $A = \begin{bmatrix} 0 & 6 & 5 & 4 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Compute A^{2013}.

Hint: Find A^n for $n \leq 4$.

Solution: We compute the first powers of A:

$$A^2 = AA = \begin{bmatrix} 0 & 0 & 18 & 17 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^3 = A^2A = \begin{bmatrix} 0 & 0 & 0 & 18 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A^4 = A^3A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, $A^n = 0$,the zero matrix for $n \geq 4$ Hence, $A^{2013} = 0$