MAT2377

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Two Sample Test 10-3.1

Assumptions:

- $X_{11}, \ldots, X_{1n_1}$ is a random sample from population 1, so the size is n_1,
- $X_{21}, \ldots, X_{2n_2}$ is a random sample from population 2, so the size is n_2,
- the two populations are independent and normal with means μ_1 and μ_2, respectively.
- **Variances are unknown**

We want to test

$$H_0 : \mu_1 = \mu_2, \qquad H_1 : \mu_1 \neq \mu_2.$$

Let

$$\bar{X}_1 = \frac{1}{n_1}\sum_{i=1}^{n_1} X_{1i}, \qquad \bar{X}_2 = \frac{1}{n_2}\sum_{i=1}^{n_2} X_{2i}.$$

The pooled variance is

$$S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2},$$

where S_1^2 and S_2^2 are sample variances for the respective samples, i.e.

$$S_1^2 = \frac{1}{n_1-1}\sum_{i=1}^{n_1}(X_{1i}-\bar{X}_1)^2, \qquad S_2^2 = \frac{1}{n_2-1}\sum_{i=1}^{n_2}(X_{2i}-\bar{X}_2)^2$$

The test statistics is

$$T_0 = \frac{\bar{X}_1 - \bar{X}_2}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1+n_2-2}.$$

Example: Two catalysts are analyzed to determine how they affect the mean yield of a chemical process. A test is run and the results are $n_1 = n_2 = 8$, $\bar{x}_1 = 92.255$, $s_1 = 2.39$, $\bar{x}_2 = 92.733$, $s_2 = 2.98$. Is there any difference between the mean yields? Use $\alpha = 0.05$.*Solution:*

We have $s_p^2 = 7.29625 \cong 7.3$. The observed value of test statistics is

$$t_0 = \frac{\bar{x}_1 - \bar{x}_2}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = -0.35$$

We have $t_{0.025,14} = 2.145$. We reject H_0 if $|t_0| > 2.145$. Don't reject.

Paired test

Assumptions:

- $X_{11}, \dots, X_{1n}$ is a random sample from population 1
- $X_{21}, \dots, X_{2n}$ is a random sample from population 2
- Two populations are not independent and normal with means μ_1 and μ_2, respectively.
- Variances are unknown

We want to test

$$H_0 : \mu_1 = \mu_2, \qquad H_1 : \mu_1 \neq \mu_2.$$

Solution: Compute differences $D_i = X_{1i} - X_{2i}$ and consider the t-test. The test statistics is

$$T_0 = \frac{\bar{D}}{S_D/\sqrt{n}} \sim t_{n-1},$$

where

$$\bar{D} = \frac{1}{n}\sum_{i=1}^{n} D_i,$$

$$S_D^2 = \frac{1}{n-1}\sum_{i=1}^{n}(D_i - \bar{D})^2,$$

the standard deviation of the differences.

So, what is the rule for rejection? This:

Rule: either $t_0 > t_{\alpha/2,n-1}$ or $t_0 < -t_{\alpha/2,n-1}$

Note: You can also consider one-sided alternatives for both two-sample and paired test.

For the last test we have discussed here, these are the rejection rules:

— $H_1 : \mu_1 > \mu_2$, $t_0 > t_{\alpha,n-1}$;

— $H_1 : \mu_1 < \mu_2$, $t_0 < -t_{\alpha,n-1}$;

Exc: Do the one-sided rejection rules for the first test done today!

Exp Consider the data:

	X_{1i}	X_{2i}	D_i
1	1.186	1.061	0.125
2	1.151	0.992	0.159
3	1.322	1.063	0.259
4	1.339	1.062	0.277
5	1.200	1.065	0.135
6	1.402	1.178	0.224
7	1.365	1.037	0.328
8	1.537	1.086	0.451
9	1.559	1.052	0.507

Determine whether there is any difference (on the average) between the 2 sets of numbers, (assume $\alpha = 0.05$).

Sol: We have from the statement $H_0 : \mu_1 = \mu_2$, $H_1 : \mu_1 \neq \mu_2$, $\alpha = 0.05$. We compute $\bar{D} = \frac{1}{9}\sum_{i=1}^{9} D_i = \frac{1}{9}\{0.125 + \cdots + 0.507\} \cong 0.2739$.

Then the observed standard deviation is $\sqrt{S_D^2} = \sqrt{\frac{1}{9-1}\sum_{i=1}^{n}(D_i - \bar{D})^2} = \sqrt{\frac{1}{8}\{(0.125 - 0.2739)^2 + \cdots + (0.507 - 0.2739)^2\}} = 0.1351$

The observed valur of the test statistic is $t_0 = \frac{\bar{D}}{s_D/\sqrt{n}} = \frac{0.2739}{0.1351/\sqrt{9}} = 6.08.$

We have $t_{0.025,8} = 2.306$. Comapare now: since $t_0 = 6.08 > t_{\alpha/2,n-1} = 2.306$ (so we are getting the observed value of the test statistic in the critical region) we reject H_0, i.e., the 2 data sets yield different means!

16-10: Cummulative Sum Control Chart

The control charts $\bar{X}$ have a major disadvantage: they are not efficient in detecting small shifts (for mean) in the process. A better tool is the Cummulative Sum Control Chart (CUSUM).

Let μ_0 be the target for the process mean (**mean quality** under control). We say that the sum:

$$C_i = \sum_{j=1}^{i}(\bar{x}_j - \mu_0) = (\bar{x}_i - \mu_0) + \sum_{j=1}^{i-1}(\bar{x}_j - \mu_0) = (\bar{x}_i - \mu_0) + C_{i-1}$$

is called the cumulative sum, where $\bar{x}_j$ is the mean of the jth sample. Cumulative sums are more effective for detecting small process shifts since each sum combines information from $several$ samples.

If the experiment is under control at the target value μ_0, then the CUSUMs should fluctuate around 0. IF the mean shifts upward to some number $\mu_1 > \mu_0$, then an upward or positive drift will develop in the CUSUMs C_i; IF the mean shifts downward to some number $\mu_1 < \mu_0$, then a downward or negative drift will develop in the CUSUMs C_i. If we observe a trend in the plotted C_is either downward or upward, then we take this as evidence that the process/experiment has shifted, and we must search for assignable causes.

Example 1: let us say the size is $n = 4$, and we compute the CUSUMs when the target mean $\mu_0 = 10$, and then one more time when $\mu_0 = 9.5$.

		$\mu_0 = 10$		$\mu_0 = 9.5$	
i	$\bar{x}_i$	$\bar{x}_i - \mu_0$	C_i	$\bar{x}_i - \mu_0$	C_i
1	9.4	-0.6	-0.6	-0.1	-0.1
2	11.1	1.1	0.5	1.6	1.5
3	9.1	-0.9	-0.4	-0.4	1.1
4	8.7	-1.3	-1.7	-0.8	0.3
5	11.6	1.6	-0.1	2.1	2.4
6	12.1	2.1	2	2.6	5
7	7.9	-2.1	-0.1	-1.6	3.4
8	10.9	0.9	0.8	1.4	4.8
9	9.5	-0.5	0.3	0	4.8
10	7.8	-2.2	-1.9	-1.7	3.1
11	10.5	0.5	-1.4	1	4.1
12	10.5	0.5	-0.9	1	5.1
13	11.4	1.4	0.5	1.9	7

For $\mu_0 = 10$ you may see the fluctuation around 0, for $\mu_1 = 9.5$ the CUSUMs are on average increasing, and this tells us that probably the true mean is bigger than 9.5, so there is a shift in the mean quality.

There is a problem: no LIMITS OF CONTROL!!! SO, we are introducing a **Tabular CUSUM**.

A change of 0.1cm in mean quality may not be so important, while a change of 1.0cm may be important. So:

Terminology: Say μ_1 is the value of the mean corresponding to the out-of-control state. The value of the shift is $\Delta = |\mu_1 - \mu_0|$, n is the size of sample; μ_0 target mean; $\bar{x}_i$ the mean of the ith sample; **reference value** $K = \frac{\Delta}{2}$.

N.B. One may also use $K = \sigma_{\bar{X}}/2$, or $\sigma/(2\sqrt{n})$.

Tabular CUSUM is (for sample i):

$S_H(i)$ — upper one-sided CUSUM;

$S_L(i)$ — lower one-sided CUSUM; where

$S_H(i) = \max[0, \bar{x}_i - (\mu_0 + K) + S_H(i-1)]$ and

$S_L(i) = \max[0, (\mu_0 - K) - \bar{x}_i + S_L(i-1)]$.

The initial values are $S_H(0) = 0 = S_L(0)$.

Warning of instability: We are saying that the process is not stable (or **out of control**) if either $S_H(i) > H$ or $S_L(i) > H$, where the constant H is the **decision interval**.

Choosing H and K

Define $H = h\sigma_{\bar{X}} = h\sigma/\sqrt{n}$ and $K = k\sigma_{\bar{X}} = k\sigma/\sqrt{n}$, where h and k are constants.

Empirical Rule: $k = 1/2$ and $h = 4$ or 5.

EXC: Suppose that $\mu_0 = 9.5$, $\sigma = 3$, $n = 4$. Use the first 5 samples in the previous example and create a chart of CUSUM when $h = 5$. Is the process out of control?

Sol: We have $K = \frac{1}{2}\frac{3}{\sqrt{4}} = 0.75$ and $H = 5 \times \frac{3}{\sqrt{4}} = 7.5$. Then we compute: $S_H(0) = 0 = S_L(0)$. Moreover,

$S_H(1) = \max[0, \bar{x}_1 - (\mu_0 + K) + S_H(0)] = \max[0, 9.4 - (9.5 + 0.75) + 0] = \max[0, -0.85] = 0$,

$S_L(1) = \max[0, (\mu_0 - K) - \bar{x}_1 + S_L(0)] = \max[0, (9.5 - 0.75) - 9.4 + 0] = \max[0, -0.65] = 0$;

$S_H(2) = \max[0, \bar{x}_2 - (\mu_0 + K) + S_H(1)] = \max[0, 11.1 - (9.5 + 0.75) + 0] = \max[0, 0.85] = 0.85$,

$S_L(2) = \max[0, (\mu_0 - K) - \bar{x}_2 + S_L(1)] = \max[0, (9.5 - 0.75) - 11.1 + 0] = 0$;

$S_H(3) = \max[0, \bar{x}_3 - (\mu_0 + K) + S_H(2)] = \max[0, 9.1 - (9.5 + 0.75) + 0.85] = 0$,

$S_L(3) = \max[0, (\mu_0 - K) - \bar{x}_3 + S_L(2)] = \max[0, (9.5 - 0.75) - 9.1 + 0] = 0$;

$S_H(4) = \max[0, \bar{x}_4 - (\mu_0 + K) + S_H(3)] = \max[0, 8.7 - (9.5 + 0.75) + 0] = 0$,

$S_L(4) = \max[0, (\mu_0 - K) - \bar{x}_4 + S_L(3)] = \max[0, (9.5 - 0.75) - 8.7 + 0] = 0.05$;

$S_H(5) = \max[0, \bar{x}_5 - (\mu_0 + K) + S_H(4)] = \max[0, 11.6 - (9.5 + 0.75) + 0] = 1.35$,

$S_L(5) = \max[0, (\mu_0 - K) - \bar{x}_5 + S_L(4)] = \max[0, (9.5 - 0.75) - 11.6 + 0.05] = 0$;

Since no value above exceeds H we conclude (based on the first 5 samples) that there is no warning of instability, the process is not out of control.

Review, statements:

Exc. 1 The average length of stay at a general hospital (on planet MathematiX) of a sample of 20 patients was 7 days with a sample standard deviation of 2 days. Assuming normality, find a 95% confidence interval for the average length of stay of a patient discharged from the hospital.

Exc. 2 Consider a random variable X with the following probability density function: $f(x) = \frac{3}{4}(1 - x^2)$ if $-1 < x < 1$, and $f(x) = 0$ elsewhere. Calculate the expected value and the variance of X.

Exc. 3 A certain company claims that they receive complaint-related phone calls at a mean rate of 2 per week. Find the probability that they receive at most 2 of these calls in the next 4 weeks. Assume Poisson!

Exc. 4 A diastolic blood pressure reading of less than 90 mm is considered normal. Assume that two-thirds of the participants in a very large study have diastolic blood pressure readings of less than 90 mm. For a random sample of 75 participants, approximate the probability that 45 or more will have normal diastolic readings.

Exc. 5 A chronic condition improves spontaneously in 45% of people. We would like to test the claim that a new medication could increase this percentage. Of 75 patients tested with the new medication, 38 improve. For $\alpha = 0.025$ what are the conclusions of the testing?

Exc. 6 The following measurements were recorded for the drying time, in hours, of a certain brand of latex paint:

3.4, 2.5, 4.8, 2.9, 3.6

2.8, 3.3, 5.6, 3.7, 2.8

The data is summarized as follows: $\sum_i x_i = 35.4$, $\sum_i x_i^2 = 133.84$.

Assuming that the measurements represent a random sample from a normal population, find a 95% confidence interval of the mean drying time of this brand.

Exc. 7 The results of a study to determine patterns of drug use in adolescents were published. Of the 4145 seventh graders in a random sample, 124 reported using alcohol on a weekly basis. Give a 90% confidence interval for the percent of weekly alcohol use among all seventh graders.

Exc. 8 The use of polymers in medicine, especially in the area of drug delivery, is one of the fastest growing areas of polymer chemistry. A study experiment different formulations used in a passive plus-delivery device. A sample was taken from one formulation on the first total-drug reading (in milligrams):

603, 534, 542, 591, 680

489, 516, 570, 592, 654

The desired result was for the mean of this measurement to exceed 550 mg. Using a level of significance of $\alpha = 5\%$, test to see if the formulation appears to be delivering the desired mean. Assume normality.

Exc. 9 Consider the random variables X and Y such that $E[X] = 0.5$, $E[Y] = 1.5$, $Var[X] = 0.25$, $Var[Y] = 35$ and the covariance is $\sigma_{XY} = -1.1$. Consider the following two random variables:

$W_1 = X + Y$ and $W_2 = 2X + 5Y$.

1. Are X and Y independent? 2. Compute the expectations and the variances of W_1 and W_2.

Exc. 10 A study used X-ray computed tomography to collect data on brain volumes for a group of patients with obsessive-compulsive disorders and a control group of healthy persons. Sample results (in mL) are given below for total brain volumes:

Obsessive-compulsive patients: $n = 10$, $\bar{x} = 1390.03$, $s = 156.84$;

Control Group: $n = 10$, $\bar{x} = 1268.41$, $s = 137.97$

Assume that the populations are normal with equal variances. At a level of significance of 5%, can we accept the hypothesis that the mean brain volume of obsessive-compulsive patients is larger than the mean brain volume of healthy persons?

Review:

Exc. 1 The average length of stay at a general hospital (on planet MathematiX) of a sample of 20 patients was 7 days with a sample standard deviation of 2 days. Assuming normality, find a 95% confidence interval for the average length of stay of a patient discharged from the hospital.

Sol: see pages 3,4 on the lecture June 2. We have $\alpha = 0.05$, and the C.I. is $\bar{x} \pm t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}$. So we get $7 \pm t_{0.025,19}\frac{2}{\sqrt{20}} = 7 \pm 2.093\frac{2}{\sqrt{20}} = [6.064, 7.936]$. Please note the word "normality". See pages 3,4!

Exc. 2 Consider a random variable X with the following probability density function: $f(x) = \frac{3}{4}(1 - x^2)$ if $-1 < x < 1$, and $f(x) = 0$ elsewhere. Calculate the expected value and the variance of X.

Sol: $E(X) = \int_{-1}^{1} x\frac{3}{4}(1-x^2)dx = \frac{3}{4}\int_{-1}^{1} x(1-x^2)dx = \frac{3}{4}\int_{-1}^{1}(x-x^3)dx = \frac{3}{4}\{\frac{x^2}{2} - \frac{x^4}{4}\}|_{-1}^{1} = 0$. Now $E(X^2) = \int_{-1}^{1} x^2\frac{3}{4}(1-x^2)dx = \frac{3}{4}\{\frac{x^3}{3} - \frac{x^5}{5}\}|_{-1}^{1} = \frac{1}{5}$.

Hence $Var(X) = \frac{1}{5} - 0^2 = \frac{1}{5}$.

Exc. 3 A certain company claims that they receive complaint-related phone calls at a mean rate of 2 per week. Find the probability that they receive at most 2 of these calls in the next 4 weeks. Assume Poisson!

Sol: If N is the rv the number of calls in 4 weeks, then N is Poisson with $\lambda = \frac{2\times 4}{1} = 8$. Hence we compute: $P(N \le 2) = e^{-8}\frac{8^0}{0!} + e^{-8}\frac{8^1}{1!} + e^{-8}\frac{8^2}{2!} = 0.0138$. Recall that if X **is Poisson with** λ, then $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$.

Exc. 4 A diastolic blood pressure reading of less than 90 mm is considered normal. Assume that two-thirds of the participants in a very large study have diastolic blood pressure readings of less than 90 mm. For a random sample of 75 participants, approximate the probability that 45 or more will have normal diastolic readings.

Sol: Let $N = \#$ among 75 with normal diastolic readings. Then N

follows a binomial with $n = 75$ and $p = 2/3$. We do have $P(N \geq 45) = 1 - P(N \leq 44) = 1 - P(N \leq 44.5)$. Now we do have: $np = 75 \times \frac{2}{3}$, and $np(1-p) = 75 \times \frac{2}{3} \times \frac{1}{3}$. Hence:

$P(N \geq 45) = 1 - \Phi(\frac{44.5 - \frac{75 \times 2}{3}}{\sqrt{75 \times \frac{2}{3} \times \frac{1}{3}}}) = 1 - \Phi(-1.35) = 1 - 0.088508 = 0.911492$. So we have used the **normal approximation to binomial**!

Exc. 5 A chronic condition improves spontaneously in 45% of people. We would like to test the claim that a new medication could increase this percentage. Of 75 patients tested with the new medication, 38 improve. For $\alpha = 0.025$ what are the conclusions of the testing?

Sol: We have $H_0 : p = 0.45$, $H_1 : p > 0.45$, $\alpha = 0.025$, $np_0 > 5$, $n(1-p_0) > 5$. The observed value if the test statistics is: $z_0 = \frac{\frac{38}{75} - 0.45}{\sqrt{0.45 \times (1 - 0.45)/75}} = 0.98644005$. IS $z_0 > z_\alpha = z_{0.025} = 1.96$? NO! So we fail to reject H_0 at this given level.

Exc. 6 The following measurements were recorded for the drying time, in hours, of a certain brand of latex paint:

3.4, 2.5, 4.8, 2.9, 3.6

2.8, 3.3, 5.6, 3.7, 2.8

The data is summarized as follows: $\sum_i x_i = 35.4$, $\sum_i x_i^2 = 133.84$.

Assuming that the measurements represent a random sample from a normal population, find a 95% confidence interval of the mean drying time of this brand.

Sol: We compute the sample mean $\bar{x} = \frac{1}{10}\sum_i x_i = 35.4 = 3.54$. The sample mean is computed as follows: $s = \sqrt{\frac{(\sum x_i^2) - n\bar{x}^2}{n-1}} = \sqrt{\frac{133.84 - (10)\times(3.54)^2}{10-1}} = 0.97320$. Then a 95% confidence interval for μ is $\bar{x} \pm t_{0.025,9}\frac{s}{\sqrt{n}} = 3.54 \pm 2.262 \times \frac{0.97320}{\sqrt{10}} = [2.84387, 4.23613]$.

Exc. 7 The results of a study to determine patterns of drug use in adolescents were published. Of the 4145 seventh graders in a random sample, 124 reported using alcohol on a weekly basis. Give a 90% confidence interval for the percent of weekly alcohol use among all seventh graders.

Sol: See our lecture on section 8-5 and recall that a 90% confidence interval is given by: $\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \frac{124}{4145} \pm z_{0.05}\sqrt{\frac{\frac{124}{4145}(1-\frac{124}{4145})}{4145}}$. Since $z_{0.05} = 1.645$ we get the following interval: $[0.025562878, 0.034268243]$.

Exc. 8 The use of polymers in medicine, especially in the area of drug delivery, is one of the fastest growing areas of polymer chemistry. A study experiment different formulations used in a passive plus-delivery device. A sample was taken from one formulation on the first total-drug reading (in milligrams):

603, 534, 542, 591, 680

489, 516, 570, 592, 654

The desired result was for the mean of this measurement to exceed 550 mg. Using a level of significance of $\alpha = 5\%$, test to see if the formulation appears to be delivering the desired mean. Assume normality.

Sol: The sample mean is $\bar{x} = \frac{1}{10}\sum_i x_i = \frac{5771}{10} = 577.1.$ The sample standard deviation is $s = \sqrt{\frac{(\sum x_i^2) - n\bar{x}^2}{n-1}} = \sqrt{\frac{(3362667)-(10)(577.1)^2}{10-1}} = 59.83579.$

We test $H_0 : \mu = 550$ against $H_1 : \mu > 550$, where $\alpha = 5\%$. The observed value of the test statistic is $t_0 = \frac{\bar{x}-550}{s/\sqrt{n}} = \frac{577.1-550}{59.83579/\sqrt{10}} = 1.43.$ QUESTION: Is t_0 greater than $t_{0.025,10-1} = t_{0.025,9} = 2.262$? ANSWER: NO! So we fail to reject H_0, i.e., we cannot conclude that $\mu > 550$.

Exc. 9 Consider the random variables X and Y such that $E[X] = 0.5$, $E[Y] = 1.5$, $Var[X] = 0.25$, $Var[Y] = 35$ and the covariance is $\sigma_{XY} =$

-1.1. Consider the following two random variables:

$W_1 = X + Y$ and $W_2 = 2X + 5Y$.

1. Are X and Y independent? 2. Compute the expectations and the variances of W_1 and W_2.

Sol: 1) since $\sigma_{XY} \neq 0$, then X and Y are not independent.

2) $E[W_1] = E[X] + E[Y] = 2.0$, $E[W_2] = 2E[X] + 5E[Y] = 8.5$, $Var[W_1] = Var[X] + Var[Y] + 2\sigma_{XY} = 33.05$ (recall chapter 5-5)! By the same section 5-5 (page 189) one gets: $Var[W_2] = Var[2X] + Var[5Y] + 2\sigma_{2X,5Y} = 2^2 Var[X] + 5^2 Var[Y] + 2(2)(5)\sigma_{X,Y} = 854$. WHY? Imagine this: $\sigma_{2X,5Y} = E(2X5Y) - E(2X)E(5Y) = 2 \times 5E(XY) - 2E(X)5E(Y) = 2 \times 5\sigma_{XY}$.

Exc. 10 A study used X-ray computed tomography to collect data on brain volumes for a group of patients with obsessive-compulsive disorders and a control group of healthy persons. Sample results (in mL) are given below for total brain volumes:

Obsessive-compulsive patients: $n = 10$, $\bar{x} = 1390.03$, $s = 156.84$;

Control Group: $n = 10$, $\bar{x} = 1268.41$, $s = 137.97$

Assume that the populations are normal with equal variances. At a level of significance of 5%, can we accept the hypothesis that the mean brain volume of obsessive-compulsive patients is larger than the mean brain volume of healthy persons?

Sol: We want to test $H_0 : \mu_1 - \mu_2 = 0$ against $H_1 : \mu_1 - \mu_2 > 0$ (i.e. $\mu_1 > \mu_2$). The test statistic is $T_0 = \frac{\bar{X}_1 - \bar{X}_2}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_1}}}$ and observed value is $t_0 = \frac{1390.03-1268.41}{147.70665\sqrt{\frac{1}{10}+\frac{1}{10}}} = 1.8412$. Since $t_0 > t_{\alpha,10+10-2} = 1.734$ we reject H_0 and accept H_1, i.e. $\mu_1 > \mu_2$. In other words (at a 5% significance level) the mean brain volume of obsessive-compulsive patients is larger than the mean brain volume of healthy persons!

Some important facts:

Probability

- Sample spaces and events
- Approaches to probability:

  Classical Approach: If can think of the outcomes as equally likely, then $P(A) = \frac{\#\text{ of outcomes in } A}{\#\text{ of total possible outcomes}}$

  Relative Frequency Approach: Consider n trials of the random experiment, then $P(A) \cong \frac{f_n(A)}{n}$, where $f_n(A) = \#$ of times that A occurs among the n trials.

- Operations on events: unions, intersections, complements
- Mutually exclusive events.
- Exhaustive events.
- The three axioms of probability: (Certainty): $P(S) = 1$

  (Positivity): $P(A) \geq 0$ for all events A

  (Additivity): if $A_1, A_2, \ldots$ are mutually exclusive events then $P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$
- The direct consequences of the axioms and the addition rules: $P(A \cup B) =$?

- Conditional Probability: The probability that A occurs given that B occurs is $P(A|B) = \frac{P(A \cap B)}{P(B)}$;

- multiplication rule

- rule of total probability

- Bayes' theorem

- Independent events

Discrete random variables

- The probability mass function (p.m.f.) f_X of the discrete random variable X is defined as $f_X(x) = P(X = x)$;

- Properties of the density of a discrete random variable: 1. $0 \leq f_X(x) \leq 1$;

  2. $\sum_x f_X(x) = 1$; i.e. total mass is 1

  3. (Computational Property): $P(X \in A) = \sum_{x \in A} f_X(x)$

- expectation, mean, variance, standard deviation

- Discrete distributions to know:

  1. binomial with parameters n and p, p.m.f., mean , variance, standard deviation;

  2. geometric with parameter p, p.m.f., mean , variance, standard deviation, see lecture 3;

  3. Poisson distribution with parameter λ, p.m.f., mean, variance and standard deviation

Continuous random variables

- prob. density function f_X of a continuous random variable X is defined as $f_X(x) = \frac{dF_X(x)}{dx}$; where F_X is the cumulative distribution function of X.

- Properties of the density of a continuous random variable:

  1. $f_X(x) \geq 0$
  2. $\int_{-\infty}^{\infty} f_X(x)dx = 1$; i.e. total mass is 1
  3. Computational Property: $P(X \in A) = \int_A f_X(x)dx$
  4. $P(X = x) = 0$, i.e. a point has no mass
  5. $P(a < X < b) = \int_a^b f_X(x)dx = F_X(b) - F_X(a)$

- expectation, mean, variance, standard deviation
- The exponential distribution with parameter λ
- The gamma distribution with parameters λ and r;
- The uniform distribution
- The normal distribution $N(\mu; \sigma^2)$:

  1. the percentiles and the table;
  2. 5-5 for expectation and variance of linear combos!
  3. Central Limit Theorem;
  4. Normal approximation of binomials (recall the correction, discrete vs continuous)

- Independent Random Variables

Other important facts:

Random Sampling

- $X_1, \ldots, X_n$ is a random sample if the variables are independent and identically distributed

- The underlying distribution (or the common distribution) is called the population

- A few common statistics, i.e. functions of a random sample,

— sample mean $\overline{X} = \frac{1}{n}\{X_1 + \cdots + X_n\}$

— sample variance $S^2 = \frac{(\sum_{i=1}^{n} X_i^2) - n(\overline{X})^2}{n-1}$

— sample standard deviation $S = \sqrt{S^2}$

— the sample proportion is $\hat{p} = x/n$, where $x = \#$ of successes among n independent Bernoulli trials.

Estimation of the population proportion p

- estimation
- sample size, error
- interval estimation (confidence interval)

Estimation of the population mean μ

- estimation

- interval estimation (confidence interval)

Hypothesis Testing: Understand the following concepts:

- null hypothesis H_0 and alternative hypothesis H_1 (research hypothesis)
- errors of type I and II
- right-tailed, left-tailed or two-tailed test
- test statistic
- critical region when testing a hypothesis on μ
- p-value of the test

- Significance level α

- reject H_0 and accept H_1, only if p-value $< \alpha$

Comparing Two Means:

- Unpaired data: We collect two independent samples, i.e from population 1 and from population 2. — Note that for the normal populations with equal variances case, we will compute the pooled sample standard deviation.

- Paired data: We collect a sample of n subjects from a population and take two measurements from each subject under different conditions. — We compute the observed differences $D_i = X_{1i} - X_{2i}$.

More exercises:

Exc. 11 Let X and S be the sample mean and sample standard deviation (respectively) for a sample of size $n = 15$ from a normal population of mean $\mu = 10$. Define the statistic $T = \frac{\overline{X}-\mu}{S/\sqrt{n}}$. Find t such that a) $P(T > t) = 0.05$; b) $P(T \leq t) = 0.10$.

Exc. 12 A random sample of 100 two-month-old babies is obtained, and the mean head circumference is found to be 40.6 cm. Assume that the population standard deviation is known to be 1.6 cm. Find a 95% confidence interval for the true mean head circumference of two-month-old babies.

Exc. 13 In attempting to control the strength of the wastes discharged into a nearby river, a paper firm has taken a number of measures. Members of the firm believe that they have reduced the oxygen- consuming power of their wastes from a previous mean of 500 (measured in permanganate in parts per million). To prove their claim, they plan to take reading on

25 consecutive days, giving a sample mean of 579.11. Assume that these readings can be treated as a random sample and that we can safely assume that the oxygen-consuming power of their wastes is normally distributed with variance 13225. (Use $\alpha = 1\%$)

(a) State the hypotheses.

(b) What is the test statistic?

(c) Give the critical region for this test.

(d) Give the p-value.

(e) Can we accept the firm's claim?

Exc. 14 A study of body characteristics and performance is conducted among master-class Olympic weight lifters. Two variables studied are $X =$

the subject's body weight, and $Y =$ his best reported clean and jerk lift. The data (in pounds) is recorded:

x	y
134	185
138	238
154	260
178	290
176	312
190	336
190	339
205	341
205	358
206	359

Calculate the estimated regression line.

Exc. 15 A certain steel bar is measured with a device which has a known precision $\sigma = 0.25$mm. Suppose we want to estimate the mean "measurement" with an error at most 0.2mm at the level of significance $\alpha = 0.05$. What sample size is required? Assume normality.

Exc. 16 It is claimed that the breaking strength of yarn used in manufacturing drapery material is **normally** distributed with mean 97 and $\sigma = 2$ psi. A random sample of nine specimens is tested and the average breaking strength is found to be $\bar{x} = 98$ psi. If $\alpha = 0.05$, the p-value and a conclusion for the appropriate one-sided test is what?

Exc. 17 A medical review board approves a mean stay in the hospital after a particular operation as 6.0 days. The board claims that the average for Medicare patients has been substantially longer that 6.0 days. To examine this claim a sample of 25 Medicare patients who have had this operation in the past year is selected. The sample mean and standard deviation (in days) respectively are 6.32 and 1.62. We want to test, if there is enough evidence to support Boards claim, i.e. we test $H_0 : \mu = 6$ against $H_1 : \mu > 6$, where μ is the mean stay length. Find the appropriate one-sided confidence interval for μ and the conclusion of the testing (use $\alpha = 0.01$).

Exc. 18 Let $X_1, X_2, \ldots, X_n$ denote a random sample from a population having mean μ and variance σ^2. Consider the following estimators of μ: $\hat{\Theta}_1 = (X_1+X_{22})/2$, $\hat{\Theta}_2 = 2X_5-X_1$. Comparing the MSE of the estimators, which estimator is better?

Exc. 19 Twenty five preliminary samples of size $n = 5$ have been taken and the following quantities have been computed: $\sum_{i=1}^{25} \bar{x}_i = 362.75$; $\sum_{i=1}^{25} r_i = 8.60$; $\sum_{i=1}^{25} s_i = 3.64$.

(a) Determine the control chart for X (from R);

(b) Determine the control chart for R;

(c) Determine the control chart for X (from S);

(d) Determine the control chart for S.

Exc. 20 On average, one person in 1 hundred will carry a certain mutant gene. 60 people are tested. What is the approximate probability that 5 or more of these people will be found to carry the gene?

Exc. 21 Let us say that on planet MathematiX some aliens are born on 2666. Suppose that X represents the weight of a (randomly chosen) alien at birth. The following data gives the weights at birth (in grams), for a sample of $n = 5$ aliens: $x_1 = 785, x_2 = 825, x_3 = 671, x_4 = 981, x_5 = 732$. Suppose that the alien weight X at birth is a normal random variable with standard deviation $\sigma = 115$g. Find a 95% confidence interval for the average alien weight μ at birth.

Exc. 22 Regression methods were used to analyze the data from a study investigating the relationship between roadway surface temperature (x) and pavement deflection (y). Summary quantities were $n = 20$,$\sum_i y_i = 12.75$, $\sum_i y_i^2 = 8.86$, $\sum_i x_i = 1478$, $\sum_i x_i^2 = 143215.8$, and $\sum_i x_i y_i = 1083.67$. Compute the sample correlation coeficient R.

Exc. 23 A machine produces metal rods used in an automobile suspension system. A random sample of 8 rods is selected, and the diameter is measured. The resulting data (in millimeters) are as follows:

8.24, 8.25, 8.20, 8.23,

8.21, 8.26, 8.26, 8.28.

Assume that the population is normally distributed. **a)** Can we reject the claim that the mean rod diameter is 8.25 mm at a level of significance of 5%? (hint: use a critical region). **b)** Calculate a 95% confidence interval for the mean rod diameter. Does the confidence interval support the conclusion from part **(a)**?

Exc. 24 A textile fiber manufacturer is investigating a new drapery yarn. The company wishes to test the hypothesis $H0 : \mu = 12$ against $H_1 : \mu < 12$, using a random sample of 16 specimens. Assume a normal population with a standard deviation of 0.5 kilograms.

(a) What is the type I error probability if the critical region is defined as $\bar{x} < 11.5$ kilograms.

(b) Find β for the case where the true mean elongation is 11.25 kilograms.

(c) Find β for the case where the true mean is 11.5 kilograms.

Solutions:

Exc. 11 Let X and S be the sample mean and sample standard deviation (respectively) for a sample of size $n = 15$ from a normal population of mean $\mu = 10$. Define the statistic $T = \frac{\overline{X}-\mu}{S/\sqrt{n}}$. Find t such that a) $P(T > t) = 0.05$; b) $P(T \leq t) = 0.10$.

Sol: We know from ch. 8 (8-3.1) that $T \sim t$ with $\nu = n - 1$ degrees of freedom. So 14 degrees of freedom; **a)** By table V one gets that $t = 1.761$; for **b)** note that $1 - P(T > t) = 0.1$ implies that $P(T > t) = 0.9$, hence $t = t_{0.9,14} = -t_{0.1,14} = -1.345$. What did we use here? This formula: $t_{1-\alpha,n-1} = -t_{\alpha,n}$.

Exc. 12 A random sample of 100 two-month-old babies is obtained, and the mean head circumference is found to be 40.6 cm. Assume that the population standard deviation is known to be 1.6 cm. Find a 95% confidence interval for the true mean head circumference of two-month-old babies.

Sol: Since $\sigma = 1.6$ is known and that the sample size $n = 100 > 30$ is large, then a 95% C.I. for the population mean μ is given by(see 8-2.1):

$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 40.6 \pm z_{0.025}\frac{1.6}{\sqrt{100}} = [40.29, 40, 91]$, a very short interval!

Exc. 13 In attempting to control the strength of the wastes discharged into a nearby river, a paper firm has taken a number of measures. Members of the firm believe that they have reduced the oxygen- consuming power of their wastes from a previous mean of 500 (measured in permanganate in parts per million). To prove their claim, they plan to take reading on 25 consecutive days, giving a sample mean of 579.11. Assume that these readings can be treated as a random sample and that we can safely assume that the oxygen-consuming power of their wastes is normally distributed with variance 13225. (Use $\alpha = 1\%$)

(a) State the hypotheses.

(b) What is the test statistic?

(c) Give the critical region for this test.

(d) Give the p-value.

(e) Can we accept the firm's claim?

Sol: a) Let μ be true mean oxygen-consuming power of their wastes measured in permanganate in parts per million. Then we will test $H_0 : \mu = 500$ against $H_1 : \mu < 500$.

b) Since the population is normal and that the population standard deviation $\sigma = \sqrt{13225}$ is known, then we can use the following test statistic: $Z_0 = \frac{\overline{X}-500}{\sqrt{13225}/\sqrt{25}} = \frac{\overline{X}-500}{23}$.

c) Since $H_1 : \mu < 500$, then this is a left-tailed test. Then, the critical region is:

we reject H_0, if $z_0 < -z_{0.01} = -2.326$, where z_0 is the observed value of the Z_0 test statistic.

d) The observed value of the Z_0 test statistic is $z_0 = \frac{579.11-500}{23} = 3.43.$

Since this is a left-tailed test, then the p-value is $p = P(Z < z_0) = \Phi(3.44) \cong 1.$ See page 15 of 2009/06/16 Lecture (ch. 9)

e) **[Using the p-value:]** Since the p-value is very large, i.e. $p > \alpha = 0.01$, we cannot reject the null hypothesis. So at a level of significance of 1%, we cannot accept the firm's claim.

[Using the critical region:] Since $z_0 \geq -2.326$, we cannot reject the null hypothesis. So at a level of significance of 1%, we cannot accept the firm's claim.

Exc. 14 A study of body characteristics and performance is conducted among master-class Olympic weight lifters. Two variables studied are $X =$ the subject's body weight, and $Y =$ his best reported clean and jerk lift. The data (in pounds) is recorded:

x	y
134	185
138	238
154	260
178	290
176	312
190	336
190	339
205	341
205	358
206	359

Calculate the estimated regression line.

Sol: $\hat{y} = 2.0631x - 64.6066$. Indeed, we do have $\sum_i x_i = 1776$, so $\bar{x} = 177.6$. Now $S_{xx} = \left[\sum_i x_i^2\right] - n(\bar{x})^2 = \cdots = 6644.4$; $\bar{y} = 301.8$; $S_{xy} =$

$\left[\sum_i x_i y_i\right] - n\bar{x}\bar{y} = 13708.2$. So $\hat{\beta}_1 = 13708.2/6644.4 = 2.063120824$ and $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x} = 301.8 - 2.063120824 \times 177.6 = -64.61025834$.

Exc. 15 A certain steel bar is measured with a device which has a known precision $\sigma = 0.25$mm. Suppose we want to estimate the mean "measurement" with an error at most 0.2mm at the level of significance $\alpha = 0.05$. What sample size is required? Assume normality.

Sol: The formula (see section 8-2/263) is: $n = \{\frac{z_{\alpha/2}\sigma}{E}\}^2 = \{\frac{1.96\times0.25}{0.2}\}^2 = 6.0025$.

Exc. 16 It is claimed that the breaking strength of yarn used in manufacturing drapery material is **normally** distributed with mean 97 and $\sigma = 2$ psi. A random sample of nine specimens is tested and the average breaking strength is found to be $\bar{x} = 98$ psi. If $\alpha = 0.05$, the p-value and a conclusion for the appropriate one-sided test is what?

Sol: We have $H_0 : \mu = 97$, and $H_1 : \mu > 97$ from the statement $(98 > 97)$!!! The observed value of the test statistics (see page 10 on lecture 2009/06/16) is $z_0 = \frac{\bar{x}-97}{\sigma/\sqrt{9}} = \frac{98-97}{2/3} = 1.5$. Since it is a right test, we have by page 15 (same lecture) that the p-value is $1 - \Phi(z_0) = 1 - 0.933193 = 0.066807$. Since the p-value is not strictly smaller than $\alpha = 0.05$ we fail to reject H_0!

Exc. 17 A medical review board approves a mean stay in the hospital after a particular operation as 6.0 days. The board claims that the average for Medicare patients has been substantially longer that 6.0 days. To examine this claim a sample of 25 Medicare patients who have had this operation in the past year is selected. The sample mean and standard deviation (in days) respectively are 6.32 and 1.62. We want to test, if there is enough evidence to support Boards claim, i.e. we test $H_0 : \mu = 6$ against $H_1 : \mu > 6$, where μ is the mean stay length. Find the appropriate one-sided confidence interval for μ and the conclusion of the testing (use

$\alpha = 0.01$).

Sol: There are 2 things asked here! FIRST: let us test! The critical region is $t_0 > t_{0.01,n-1} = t_{0.01,24} = 2.492$. Now the observed value of the test statistics is $t_0 = \frac{\bar{x}-6}{1.62/\sqrt{25}} = \frac{6.32-6}{1.62/5} = 0.987654321$. WE FAIL to reject H_0.

Second part: the C.I. (by ch. 8, or lecture given on June 2) is $(\bar{x} - t_{0.01,n-1}\frac{s}{\sqrt{n}}, \infty) = (5.51, \infty)$.

Exc. 18 Let $X_1, X_2, \ldots, X_n$ denote a random sample from a population having mean μ and variance σ^2. Consider the following estimators of μ: $\hat{\Theta}_1 = (X_1 + X_{22})/2$, $\hat{\Theta}_2 = 2X_5 - X_1$. Comparing the MSE of the estimators, which estimator is better?

Sol: note first that they are unbiased! Indeed, $E(\hat{\Theta}_1) = (\mu + \mu)/2 = \mu$ and $E(\hat{\Theta}_2) = 2\mu - \mu = \mu$. So we just need to compare the variances.

Note that $Var((X_1 + X_{22})/2) = \frac{1}{4}\sigma^2 + \frac{1}{4}\sigma^2 = \frac{1}{2}\sigma^2$ and observe that $Var(2X_5 - X_1) = 4\sigma^2 + \sigma^2 = 5\sigma^2$. Since $5\sigma^2 > \frac{1}{2}\sigma^2$ we conclude that $\hat{\Theta}_1$ is better.

Exc. 19 Twenty five preliminary samples of size $n = 5$ have been taken and the following quantities have been computed: $\sum_{i=1}^{25} \bar{x}_i = 362.75$; $\sum_{i=1}^{25} r_i = 8.60$; $\sum_{i=1}^{25} s_i = 3.64$.

(a) Determine the control chart for X (from R);

(b) Determine the control chart for R;

(c) Determine the control chart for X (from S);

(d) Determine the control chart for S.

Sol: a) We get $\bar{\bar{x}} = 362.75/25 = 14.51$; $\bar{r} = 8.60/25 = 0.344$ and from page 732, table XI one gets $A_2 = 0.577$. So we get:

$CL = \bar{\bar{x}} = 14.51$; $LCL = \bar{\bar{x}} - A_2\bar{r} = 14.51 - (0.577)0.344 \cong 14.312$; $UCL = \bar{\bar{x}} + A_2\bar{r} = 14.51 + (0.577)0.344 \cong 14.708$;

b) Recal that $CL = \bar{r} = 0.344$; $LCL = D_3\bar{r} = 0$; $UCL = D_4\bar{r} = 2.115 \times 0.344 \cong 0.728$;

c) We get that $\bar{s} = 3.64/25 = 0.1456$. So we get $CL = \bar{\bar{x}} = 14.51$; $LCL = \bar{\bar{x}} - \frac{3}{c_4\sqrt{n}}\bar{s} = 14.51 - \frac{3}{0.94\sqrt{5}}0.1456 = 14.302$, and $LCL = \bar{\bar{x}} + \frac{3}{c_4\sqrt{n}}\bar{s} = 14.51 + \frac{3}{0.94\sqrt{5}}0.1456 = 14.718$.

d) Recal that $CL = \bar{s} = 0.1456$; $LCL = \bar{s}\{1 - \frac{3}{c_4}\sqrt{1 - c_4^2}\} < 0$, so we take $LCL = 0$; $UCL = \bar{s}\{1 + \frac{3}{c_4}\sqrt{1 - c_4^2}\} = 0.1456 + 3\frac{0.1456}{0.94}\sqrt{1 - (0.94)^2} = 0.304$.

Exc. 20 On average, one person in 1 hundred will carry a certain mutant gene. 60 people are tested. What is the approximate probability that 5 or more of these people will be found to carry the gene?

Sol: Let X = number of persons which carry the gene. Then $X \sim B(60, 0.01)$. Note that we are asked to compute $P(X \geq 5) = 1 - P(X \leq 4)$. We have $E(X) = np = 60 \times 0.01 = 0.6$ and $Var(X) = np(1-p) = 0.6 \times 0.99 = 0.594$. Since n is large we must use the continuity correction:

$P(X \leq 4) = P(Z \leq \frac{4+0.5-0.6}{\sqrt{0.594}}) = P(Z \leq 5.060243) = 1$, so $P(X \geq 5) = 1 - 1 = 0$.

Exc. 21 Let us say that on planet MathematiX some aliens are born on 2666. Suppose that X represents the weight of a (randomly chosen) alien at birth. The following data gives the weights at birth (in grams), for a sample of $n = 5$ aliens: $x_1 = 785, x_2 = 825, x_3 = 671, x_4 = 981, x_5 = 732$. Suppose that the alien weight X at birth is a normal random variable with standard deviation $\sigma = 115$g. Find a 95% confidence interval for the average alien weight μ at birth.

Sol: Note that $\bar{x} = \frac{1}{5}\{785 + 825 + 671 + 981 + 732\} = 798.8$g. The C.I.

is given by $798.8 \pm 1.96\frac{115}{\sqrt{5}} = [688.0, 899.6]$. So: with probability 95% the average alien weight lies between 688.0g and 899.6g on planet MathematiX!

Exc. 22 Regression methods were used to analyze the data from a study investigating the relationship between roadway surface temperature (x) and pavement deflection (y). Summary quantities were $n = 20$,$\sum_i y_i = 12.75$, $\sum_i y_i^2 = 8.86$, $\sum_i x_i = 1478$, $\sum_i x_i^2 = 143215.8$, and $\sum_i x_i y_i = 1083.67$. Compute the sample correlation coeficient R.

Sol: Recall that $R = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$. Recall from ex 11-2 solved in the slides that $S_{xy} = 141.445$, $S_{xx} = 33991.6$ and $S_{yy} = 0.731875$. So $R = \frac{141.445}{\sqrt{33991.6}\sqrt{0.731875}} \cong 0.897$.

Exc. 23 A machine produces metal rods used in an automobile suspension system. A random sample of 8 rods is selected, and the diameter is measured. The resulting data (in millimeters) are as follows:

8.24, 8.25, 8.20, 8.23,

8.21, 8.26, 8.26, 8.28.

Assume that the population is normally distributed. **a)** *Can we reject the claim that the mean rod diameter is* 8.25 mm at a level of significance of 5%? (hint: use a critical region). **b)** Calculate a 95% confidence interval for the mean rod diameter. Does the confidence interval *support* the conclusion from part **(a)**?

Sol: **a)** Note that we have from the date presented above the following computtaions: $\sum_i x_i = 65.93$, $\sum_i x_i^2 = 543.3507$. So $\bar{x} = 65.93/8 = 8.24125$; $s = \sqrt{\frac{(\sum_i x_i^2) - n\bar{x}^2}{n-1}} = 0.02696$. Now we would like to test $H_0 : \mu = 8.25$ against $H_1 : \mu \neq 8.25$ (See the statement to see why $\neq$ is chosen). Next note that the population is normal and σ is unknown, so we will use the following test statistic $T_0 = \frac{\bar{X} - 8.25}{S/\sqrt{n}}$. Then we compute the observed value: $t_0 = -0.918$. It is a two-tailed (2-sided) test, so the critical region is given by: $t_0 < -t_{0.025,7} = -2.365$ OR $t_0 > t_{0.025,7} = 2.365$ by table V. Since $-2.365 < t_0 < 2.365$, then we fail to reject H_0. Thus, we cannot reject the claim that the mean rod diameter is 8.25mm at a level of significance of 5%.

b) A 95% confidence interval for μ is given by $\bar{x} \pm t_{0.025,7}\frac{s}{\sqrt{n}} = 8.24125 \pm 2.365 \times \frac{0.02696}{\sqrt{8}} = 8.24125 \pm 0.02254 = [8.22, 8.26]$. Since 8.25 is in the interval, then we cannot conclude that $\mu \neq 8.25$. This is consistent with the conclusion in **(a)**.

Exc. 24 A textile fiber manufacturer is investigating a new drapery yarn. The company wishes to test the hypothesis $H0 : \mu = 12$ against $H_1 : \mu < 12$, using a random sample of 16 specimens. Assume a normal population with a standard deviation of 0.5 kilograms.

(a) What is the type I error probability if the critical region is defined as

$\bar{x} < 11.5$ kilograms.

(b) Find β for the case where the true mean elongation is 11.25 kilograms.

(c) Find β for the case where the true mean is 11.5 kilograms.

Sol: **(a)** $\alpha = P(type \quad I \quad error) =$

$P(reject \quad H_0 \quad when \quad H_0 \quad is \quad true) = P(\bar{X} < 11.5|\mu = 12) = \Phi(\frac{11.5-12}{0.5/\sqrt{16}}) = \Phi(-4) \cong 0.$

(b) $\beta = P(type \quad II \quad error \quad when \quad \mu = 11.25)$

$= P(do \quad not \quad reject \quad H_0 \quad when \quad \mu = 11.25) = P(\bar{X} \geq 11.5|\mu = 11.25) = 1 - \Phi(\frac{11.5-11.25}{0.5/\sqrt{16}}) = 1 - \Phi(2) = 1 - 0.977250 = 0.02275;$

(c) $\beta = P(type \quad II \quad error \quad when \quad \mu = 11.5)$

$= P(do \quad not \quad reject \quad H_0 \quad when \quad \mu = 11.5) = P(\bar{X} \geq 11.5|\mu = 11.5) = 1 - \Phi(\frac{11.5-11.5}{0.5/\sqrt{16}}) = 1 - \Phi(0) = 1 - 0.5 = 0.5$