MAT2377

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Summary of Lecture 1

- Sample spaces and events; $A \cap B$, $A \cup B$
- Principles of counting: sampling with/without replacement, ordered/unordered samples; $\underbrace{n\, n \cdots n}_{r \text{ factors}} = n^r$, C_n^r, P_n^r

Classical Probability

For situations where we have a random experiment which has exactly c possible mutually exclusive, equally likely, simple outcomes we can assign a probability to an event A by counting the number of simple outcomes that correspond to A. If the count is a then

$$P(A) = \frac{a}{c}.$$

▷ if a sample space consists of N equally likely events, the probability of each outcome is $1/N$.

Example 1:

1. Toss a fair coin. The sample space is $\mathcal{S} = \{\text{Head, Tail}\}$ The probability of observing a Head is $\frac{1}{2}$.

2. Throw a fair six sided die. There are 6 possible outcomes.

$$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}.$$

If A corresponds to observing a multiple of 3 then, in set notation

$$A = \{3, 6\}.$$

$$\text{Prob(number is a multiple of 3)} = P(A) = \frac{2}{6} = \frac{1}{3}.$$

Furthermore:

- Prob$\{$even no.$\} = P(\{2,4,6\}) = \frac{3}{6} = \frac{1}{2}$.

- Prob$\{$prime no.$\} = P(\{2,3,5\}) = 1 - P(\{1,4,6\}) = \frac{1}{2}$.

3. In a group of 1000 people it is known that 545 have high blood pressure. 1 person is selected randomly. What is the probability that this person has high blood pressure?

Answer: Relative frequency of people with high blood pressure is $\frac{545}{1000}$. Via the classical definition, this is the probability we are looking for.

Axioms of probability

1. For any event A, $1 \geq P(A) \geq 0$.

2. For the complete sample space $\mathcal{S}$, $P(\mathcal{S}) = 1$.

3. For two mutually exclusive events A and B the probability that A or B occurs is $P(A \cup B) = P(A) + P(B)$.

Since $\mathcal{S} = A \cup A^c$, and A, A^c mut. excl., $1 \overset{\textbf{A2}}{=} P(\mathcal{S}) = P(A \cup A^c) \overset{\textbf{A3}}{=} P(A) + P(A^c)$. Thus

$$P(A^c) = 1 - P(A)$$

Example 2: 1. Throw a single six sided die. Let $A = \{3, 6\}$ - the number is a multiple of 3, $B = \{1, 2\}$ - the number is less than 3. A and B are mutually exclusive.

$$P(A \text{ or } B \text{ occurs}) = P(A) + P(B) = \frac{2}{6} + \frac{2}{6} = \frac{2}{3}.$$

2. I have an urn containing 4 white balls, 3 red balls and 1 blue ball. I draw one ball and note the events $W = \{\text{the ball is white}\}$, $R = \{\text{the ball is red}\}$ and $B = \{\text{the ball is blue}\}$. Then

$$P(W) = 4/8 = 1/2, \quad P(R) = 3/8, \quad P(B) = 1/8.$$

Also, the probability of drawing a white or a red is

$$P(W \text{ or } R) = P(W \cup R) = 7/8.$$

General addition rule

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)\,.$$

Example 3: An electronic assembly consists of two components A and B. Experience tells us that $P\{A \text{ fails}\} = 0.2$, $P\{B \text{ fails}\} = 0.3$ and $P\{\text{both } A \text{ and } B \text{ fail}\} = 0.15$.

Find $P\{\text{at least one of } A \text{ and } B \text{ fails}\}$ and $P\{\text{neither } A \text{ nor } B \text{ fails}\}$. Write A for "A fails" and similarly for B. Then we want

$$\begin{aligned} P\{\text{at least one fails}\} &= P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.35\,; \\ P\{\text{neither fail}\} &= 1 - P\{\text{at least one fails}\} = 0.65\,. \end{aligned}$$

Conditional probability and independent events

Any two events A and B satisfying

$$P(A \cap B) = P(A)P(B)$$

are said to be **independent**; this is a purely mathematical definition, but agrees with intuitive notion in simple examples.

- Flip a coin twice, so 4 possible outcomes are equally likely: let $A = \{HH\} \cup \{HT\}$ denote "head on first flip", $B = \{HH\} \cup \{TH\}$ "head on second flip". Then (since $= \frac{1}{4} + \frac{1}{4}$)

$$P(A) = P(HH) + P(HT) = \frac{1}{2} \quad , \quad P(B) = P(HH) + P(TH) = \frac{1}{2}.$$

But $P(A \cap B) = P(HH) = \frac{1}{4} = P(A)P(B)$, so A and B are independent.

- A card is drawn from a well shuffled deck. Let A be the event that it is an ace and let D be the event that it is a diamond.

  We compute: $P(A) = \frac{4}{52} = \frac{1}{13}$ and $P(D) = \frac{13}{52} = \frac{1}{4}$. Also $P(A \cap D) = \frac{1}{52} = P(A)P(D)$; so independent!

- A system fails if both a component and a backup component fail.If the failures of each of the two components are independent events with probability p, then the chance of failure of the system is p^2.

- A six-sided die numbered 1–6 is loaded in such a way that the prob of getting each value is *proportional* to that value. Find prob. of a 3.

For some value v, $P(1) = v$, . . . , $P(6) = 6v$. Since these add to 1,

$$1 \quad = \quad v + 2v + 3v + 4v + 5v + 6v = 21v\,.$$

Hence $v = 1/21$, $P(3) = 3v = 3/21 = 1/7$. Now let us say that the die is rolled twice *independently*. Find the prob of getting two 3's.

If experiment is such that $P(\text{3 on Ist}) = 1/7$ and $P(\text{3 on 2nd}) = 1/7$ **and** the two events are independent, then

$$P\left(\{\text{3 on 1st}\} \cap \{\text{3 on 2nd}\}\right) = P(\text{3 on 1st})P(\text{3 on 2nd}) = 1/49\,.$$

- Which plane is more likely to crash: 2- or 3- engines one?

  $Answer:$ Assumptions - engines fail independently. We assume that the probability that the engine fails is p. A 2 engined plane will crash iff both engines fail - probability p^2. A 3 engined plane will crash iff any pair of engines fail, or if all 3 fail together. The probability of a pair of engines failing is $p * p * (1 - p)$: i.e. FAIL FAIL OK. There are 3 DIFFERENT pairs to be considered: AB, BC, or AC. The probability of all three engines failing is p^3. Therefore the probability of at least 2 engines failing is: $3p^2(1 - p) + p^3 = 3p^2 - 2p^3$. So, what does all this mean? Well, basically it's safer to use a 2-engined plane than a 3-engined plane: the 3-engined plane will crash more often, assuming that it needs 2 engines to fly.

You can sort of make sense of this by thinking that the 2-engine plane needs 50% of its engines working, while the 3-engine plane needs 66%. Of course, you could always travel by Greyhound. WHAT is important is this:$p^2 \leq 3p^2 - 2p^3$ because $0 \leq p \leq 1$.

More than two events

Three events A, B and C are **pairwise independent** if

$$P(A \cap B) = P(A)P(B),\ P(A \cap C) = P(A)P(C),\ P(B \cap C) = P(B)P(C)$$

but they aren't (fully) independent unless in addition

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

Similar higher-order equalities need to hold for four or more.

Conditional Probability

We can better understand independence by defining the conditional probability of $\boldsymbol{A}$ given $\boldsymbol{B}$ as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Note that this only makes sense when "B can happen" i.e. $P(B) > 0$.

This is effectively making B the whole sample space, "blowing up" the weight of each outcome in B so they add to 1.

Example 6: From a group of 100 people, 1 is selected. What is the probability that this person has high blood pressure? This is (unconditional) probability.

Now, from this group we select first all people with high cholesterol level, and then from the latter group we select 1 person. What is the probability that this person has high blood pressure? This is conditional probability; the probability of selecting a person with high blood pressure, given high cholesterol level.

Example 7: A sample of 249 Eskimos was taken and each person classified by blood group and tuberculosis(TB) status.

	O	**A**	**B**	**AB**	Total
TB	34	37	31	11	113
No TB	55	50	24	7	136
Total	89	87	55	18	249

The rel. frequency of people with TB is $\frac{113}{249} = 0.454$. In other words, the prob. that randomly selected person has TB is 0.454.

Of those with type **B** blood the rel. frequency of TB is

$$\text{rel.fr}(\text{TB}|\text{type } \mathbf{B}) = \frac{31}{55} = \frac{31/249}{55/249} = 0.564.$$

Example :

- Suppose we flip a coin three times. The sample space is $\Omega = \{TTT, HTT, THT, TTH, HHT, HTH, THH, HHH\}$. Consider
  - $A = \{\text{1 head}\} = \{HTT, THT, TTH\}$;
  - $B = \{\text{head on 1st}\} = \{HHT, HTH, HTT, HHH\}$;
  - $C = \{\text{first two flips the same}\} = \{TTT, TTH, HHT, HHH\}$.

- If all outcomes equally likely, are any pairs independent?
  - $P(A) = 3/8$, $P(B) = 4/8 = 1/2$, $P(C) = 4/8 = 1/2$.
  - $A \cap B = \{HTT\}$, $P(A \cap B) = 1/8 \neq P(A)P(B)$, A, B not indep.
  - $A \cap C = \{TTH\}$, $P(A \cap C) = 1/8 \neq P(A)P(C)$, A, C not indep.
  - $B \cap C = \{HHT, HHH\}$, $P(B \cap C) = 1/4 = P(B)P(C)$, indep.

- Compute four conditional probabilities $P(A|C)$, $P(A|C^c)$, $P(B|C)$, $P(B|C^c)$.

  - $C^c = \{\text{first two diff.}\} = \{HTT, HTH, THT, THH\}$, $P(C^c) = \frac{1}{2}$.
  - $A \cap C^c = \{HTT, THT\}$, $P(A \cap C^c) = 2/8 = 1/4$.
  - $B \cap C^c = \{HTH, HTT\}$, $P(B \cap C^c) = 2/8 = 1/4$.

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{1/8}{4/8} = \frac{1}{4} \quad \neq \quad P(A|C^c) = \frac{P(A \cap C^c)}{P(C^c)} = \frac{1/4}{4/8} = \frac{1}{2}.$$

$$P(B|C) = \frac{P(B \cap C)}{P(C)} = \frac{1/4}{4/8} = \frac{1}{2} \quad = \quad P(B|C^c) = \frac{P(B \cap C^c)}{P(C^c)} = \frac{1/4}{4/8} = \frac{1}{2}.$$

- $P(A|C) \neq P(A|C^c)$, A, C not indep.;
  $P(B|C) = P(B|C^c) = P(B)$, B, C indep.

In general we have **Theorem:** Two events A and B are independent iff one of the following equivalent statements is true:

$$P(A|B) = P(A)$$

$$P(B|A) = P(B)$$

$$P(A) = P(A)P(B)$$

Total Probability Rule

If $A_1, ...A_k$ are mutually exclusive and exhaustive (i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$ and $A_1 \cup .... \cup A_k = S$), then for any event B

$$P(B) = P(B \mid A_1)P(A_1) + ... + P(B \mid A_k)P(A_k)$$

Easy proof: $B = B \cap S = B \cap (A_1 \cup .... \cup A_k) = (B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_k)$ and now just apply the probability (mut exclus....)!

After an experiment generates an outcome, we are interested in the probability that a condition was present given an outcome.

Bayes' Theorem: If $A_1, ...A_k$ are mutually exclusive and exhaustive (i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$ and $A_1 \cup .... \cup A_k = S$), then for any event B and for each i,

$$P(A_i \mid B) = \frac{P(B \mid A_i)P(A_i)}{P(B)} = \frac{P(B \mid A_i)P(A_i)}{P(B \mid A_1)P(A_1) + ... + P(B \mid A_k)P(A_k)}$$

Example 9: Nissan sold three models of cars in North America in 1999: Sentras, Maximas and Pathfinders. Of the vehicles sold, 50% were Sentras, 30% were Maximas and 20% were Pathfinders. In the same year 12% of the Sentras, 15% of the Maximas and 25% of the Pathfinders had a defect.

1. I own a 1999 Nissan. What is the probability that it has the defect?

2. My 1999 Nissan has the defect. What model do you think I own? Why?
   Answer: In the first part we want to compute (with the help of the total probability rule) $P(D)$:

$$\begin{aligned} P(D) &= P(D|S)P(S) + P(D|M)P(M) + P(D|Pa)P(Pa) \\ &= 0.12 * 0.5 + 0.15 * 0.3 + 0.12 * 0.2 = etc = 0.534 \end{aligned}$$

In the second part we compare $P(M|D)$, $P(S|D)$, $P(Pa|D)$, so we are using Bayes' Theorem:

$$P(M|D) = \frac{P(D|M)P(M)}{P(D)} = \frac{0.15 \times 0.3}{0.534} = etc...$$

One more problem: Let us say that in the 2457 presidential election on planet MathematiX, exit polls provided the following results:

	B	**K**
no university degree (62%)	50	50
university degree (38%)	53	46

If a randomly selected respondent voted for $\mathbf{B}$, what is the probablility that the person has a university degree?

SOL: What we know is: $P(NU) = 0.62$, $P(U) = 0.38$, $P(B|NU) = 0.5$, $P(K|NU) = 0.5$, $P(B|U) = 0.53$, $P(K|U) = 0.46$. What we want is: $P(U|B) =?$ Note that Bayes' Theorem gives us: $P(U|B) = \frac{P(U \cap B)}{P(B)} = \frac{P(B|U)P(U)}{P(B)} = \frac{0.53 \times 0.38}{P(B|NU)P(NU) + P(B|U)P(U)} = \frac{0.53 \times 0.38}{(0.5 \times 0.62) + (0.53 \times 0.38)} = 39.3821\%$.