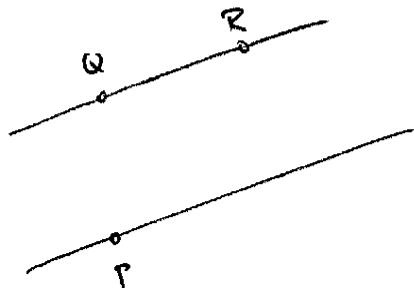


1. Find the scalar equations of the line passing through $P(2, -5, 7)$ and parallel to the line through $Q(-1, 1, -1)$ and $R(1, -1, 1)$.

- A. $x = 9 + 2t, y = -7 - 2t, z = 4 + 2t, t$ scalar.
- B. $x = -2 + 2t, y = -5 - 2t, z = 5 + 2t, t$ scalar.
- C. $x = 2 + 3t, y = -3 - 2t, z = 7 + 2t, t$ scalar.
- D. $x = 2 + 2t, y = -5 - 2t, z = 7 + 2t, t$ scalar.
- E. $x = 2 + 2t, y = -5 - 2t, z = 7 + 2t, t$ scalar.
- F. $x = 2 + t, y = -5 + 9t, z = 1 + t, t$ scalar.

My answer: D



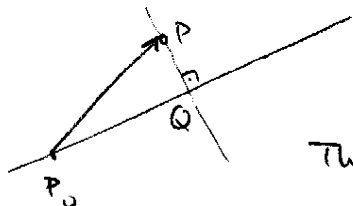
Parallel lines have the same direction vector. The direction vector of the line through Q and R is $\vec{QR} = [1, -1, 1]^T - [-1, 1, -1]^T = [2, -2, 2]^T$. Hence scalar equations of the line through $P(2, -5, 7)$ with direction vector $[2, -2, 2]^T$ are

$$x = 2 + 2t, y = -5 - 2t, z = 7 + 2t, t \text{ scalar}$$

2. Find the direction vector of the line passing through $P(3, -3, 3)$, intersecting the line $[x \ y \ z]^T = [9 \ -5 \ 7]^T + t[-1 \ 2 \ -1]^T$ and perpendicular to it.

- A. $[-1 \ 2 \ -1]^T$
- B. $[9 \ -5 \ 7]^T$
- C. Such a line does not exist.
- D. $[0 \ -4 \ -9]^T$
- E. $[-11 \ 10 \ -5]^T$
- F. $[1 \ 4 \ 7]^T$.

My answer: _____



The point $P_0(9, -5, 7)$ lies on the given line.

The vector $\vec{P_0P}$ is $[3, -3, 3]^T - [9, -5, 7]^T = [-6, 2, -4]^T$

$\vec{P_0Q} = \text{proj}_{\vec{d}}(\vec{P_0P})$ for $\vec{d} = [-1, 2, -1]^T =$ direction vector of

the given line. Hence $\vec{P_0Q} = \frac{1}{6} [-6, 2, -4]^T \cdot [-1, 2, -1]^T (\lambda) = \frac{1}{6} (6 + 4 + 4) \vec{d} = \frac{14}{6} [-1, 2, -1]^T = \frac{7}{3} [-1, 2, -1]^T$.

Now $\vec{OQ} = \vec{OP_0} + \vec{P_0Q} = [9, -5, 7]^T + \frac{7}{3} [-1, 2, -1]^T = [\frac{20}{3}, -\frac{1}{3}, \frac{14}{3}]^T$

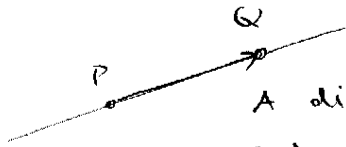
Hence the direction vector of the line we are looking for is

$\vec{QP} = [3, -3, 3]^T - [\frac{20}{3}, -\frac{1}{3}, \frac{14}{3}]^T = [-\frac{11}{3}, \frac{10}{3}, -\frac{5}{3}]^T$, or any scalar multiple of it

3. Find the scalar equation of the line passing through $P(3, -9, 0)$ and $Q(-1, 2, 7)$.

- A. $x = 3 + 4t, y = -9 + 11t, z = 7t, t$ scalar.
 B. $x = 3 - 4t, y = -9 + 11t, z = 7t, t$ scalar.
 C. $x = 1 - 3t, y = 4 + 11t, z = 1 + 2t, t$ scalar.
 D. $x = 1 + 2t, y = 1 + 4t, z = 2t, t$ scalar.
 E. $x = 3 + 4t, y = -4 + t, z = 1 + 7t, t$ scalar.
 F. $x = -3 + 4t, y = 9 + 11t, z = 7t, t$ scalar.

My answer: B



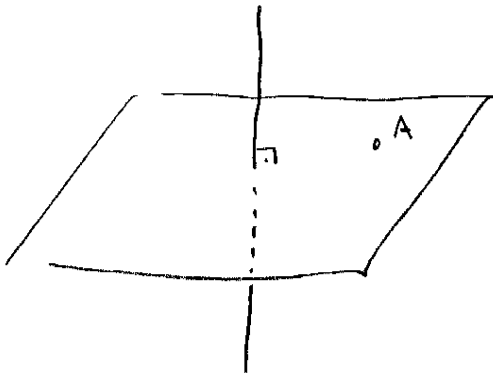
A direction vector of the line passing through P and Q is
 $\vec{PQ} = [-1, 2, 7]^T - [3, -9, 0]^T = [-4, 11, 7]^T$. Scalar equations
 are therefore as follows

$$x = 3 - 4t, \quad y = -9 + 11t, \quad z = 7t$$

4. Find the equation of the plane passing through $A(1, 0, -7)$ and perpendicular to the line $[x \ y \ z]^T = [5 \ 0 \ 5]^T + t[2 \ -1 \ 7]^T$.

- A. $2x + 3y + 6z = -1$
 B. $x - 2y + 4z = 2$
 C. $2x - y + 7z = -47$
 D. $4x - y + 2z = -4$
 E. $2x + y + z = -41$
 F. $6x - y + 2z = -40$

My answer: C



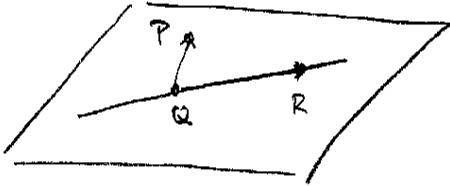
Any direction vector of the line is a normal to the plane. Hence $[2, -1, 7]^T$ is a normal, and thus an equation of the plane is

$$\begin{aligned} 2x - y + 7z &= [2, -1, 7]^T \cdot [1, 0, -7]^T \\ &= 2 + 0 - 49 = -47 \end{aligned}$$

5. Find the equation of the plane containing the point $P(1, 2, -3)$ and also containing the line through $Q(2, 0, 1)$ with direction vector $[1 \ 1 \ 1]^T$.

- A. $2x - y - z = 3$
 B. $2x - y - z = 0$
 C. $x + y + z = 3$
 D. $x + y + z = 0$
 E. $6x + 3y + z = 9$
 F. $6x + 3y + z = 0$

My answer: A



The vector $\vec{QP} = [1, 2, -3]^T - [2, 0, 1]^T = [-1, 2, -4]^T$ lies in the plane, as does the direction vector $[1, 1, 1]^T$ of the line. A normal of the plane

is therefore $[-1, 2, -4]^T \times [1, 1, 1]^T = \begin{vmatrix} i & j & k \\ -1 & 2 & -4 \\ 1 & 1 & 1 \end{vmatrix} = (|2 \ -4|, -|-1 \ -4|, |-1 \ 2|)$

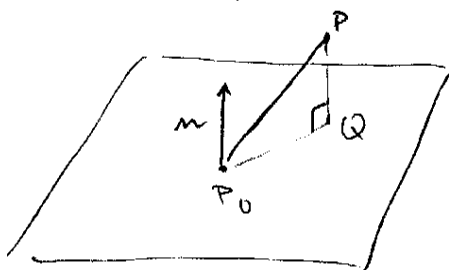
$= [6, -1-4, -1-2]^T = [6, -3, -3]^T = 3[2, -1, -1]^T$. Hence the plane has the equation $2x - y - z = c$ where the constant c is determined by the condition that $P(1, 2, -3)$ lies on the plane: $c = 2 \cdot 1 - 2 - (-3) = 3$.

Thus $2x - y - z = 3$ is an equation of the plane.

6. What is the distance between the point $P(1, 1, 1)$ and the plane $2x - 2y + z = 0$?

- A. $\frac{1}{3}$
 B. $\frac{1}{9}$
 C. $\frac{1}{\sqrt{3}}$
 D. 9
 E. 3
 F. $\sqrt{3}$

My answer: A



We follow the 1st approach of Example 12 in §3.3, and choose a point in the plane: $P_0(0, 0, 0)$. The projection of $\vec{P_0P} = \vec{OP} = [1, 1, 1]^T$ onto a normal \vec{n} of the plane is, taking $\vec{n} = [2, -2, 1]$,

$$\frac{\vec{OP} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n} = \frac{2 - 2 + 1}{4 + 4 + 1} \vec{n} = \frac{1}{5} \vec{n}.$$

The distance is $\|\text{proj}_{\vec{n}}(\vec{P_0P})\| = \|\frac{1}{5} \vec{n}\| = \frac{1}{5} \sqrt{4 + 4 + 1} = \frac{\sqrt{9}}{5} = \frac{3}{5} = \frac{1}{3}$

7. Let $u = [1 \ 1 \ 1]^T$ and $v = [2 \ -2 \ 1]^T$. Suppose w and x are vectors satisfying the following three conditions:

- $u = w + x$,
- w is a scalar multiple of v , i.e., $w = sv$ for some scalar $s \in \mathbb{R}$,
- x is orthogonal to v .

Then x is

- A. $\frac{1}{9}(7, 11, 8)$
 B. $-\frac{1}{3}(7, 11, 8)$
 C. $\frac{1}{9}(1, -1, -4)$
 D. $-\frac{1}{3}(1, -1, -4)$
 E. $\frac{1}{9}(8, 10, 4)$
 F. $-\frac{1}{3}(8, 10, 4)$.

We have $x = u - w = u - sv$ and

My answer: A

$$0 = x \cdot v = (u - sv) \cdot v = u \cdot v - s v \cdot v, \text{ hence } s = \frac{u \cdot v}{v \cdot v} =$$

$$= \frac{[1 \ 1 \ 1] \cdot [2 \ -2 \ 1]}{4 + 4 + 1} = \frac{1}{9} (2 - 2 + 1) = \frac{1}{9}. \text{ It then follows that}$$

$$x = u - \frac{1}{9}v = [1, 1, 1]^T - \frac{1}{9}[2, -2, 1]^T = \left[\frac{7}{9}, \frac{11}{9}, \frac{8}{9}\right]^T = \frac{1}{9}[7, 11, 8]$$

8. Find the area of the triangle with vertices $A(2, 0, 1)$, $B(0, 1, 2)$ and $C(1, 1, -1)$.

- A. $\frac{1}{2}\sqrt{35}$
 B. $\sqrt{35}$
 C. 25
 D. 25/2
 E. 5/2
 F. $\frac{1}{2}\sqrt{5}$

We have $\vec{AB} = [0, 1, 2]^T - [2, 0, 1]^T = [-2, 1, 1]^T$,

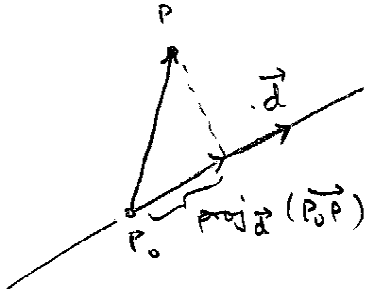
My answer: A

$\vec{AC} = [1, 1, -1]^T - [2, 0, 1]^T = [-1, 1, -2]^T$. We calculate

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -2 & 1 & 1 \\ -1 & 1 & -2 \end{vmatrix} = (| \begin{smallmatrix} 1 & 1 \\ -1 & -2 \end{smallmatrix} |, -| \begin{smallmatrix} -2 & 1 \\ -1 & -2 \end{smallmatrix} |, | \begin{smallmatrix} -2 & 1 \\ -1 & 1 \end{smallmatrix} |) = [-3, -5, -1]^T$$

The area of the triangle is $\frac{1}{2} \|\vec{AB} \times \vec{AC}\| = \frac{1}{2} \sqrt{9 + 25 + 1} = \frac{1}{2} \sqrt{35}$

9. Find the distance from the point $P(1,0,1)$ to the line through $P_0(0,0,0)$ with direction vector $d = [1 \ 1 \ 1]^T$.
- A. $\frac{2}{3}$
 B. $2\sqrt{5}$
 C. 0
 D. $\frac{1}{4}$
 E. $\frac{1}{7}\sqrt{5}$
 F. $\frac{1}{3}\sqrt{6}$

My answer: F

$$\begin{aligned} \vec{P_0P} &= \vec{OP} = [1, 0, 1]^T \\ \text{proj}_d(\vec{P_0P}) &= \frac{[1, 0, 1]^T \cdot [1, 1, 1]^T}{1+1+1} \vec{d} = \frac{2}{3} [1, 1, 1]^T \\ \text{distance} &= \|\vec{P_0P} - \text{proj}_d(\vec{P_0P})\| = \\ &= \|[1, 0, 1]^T - \frac{2}{3}[1, 1, 1]^T\| = \|[1/3, -2/3, 1/3]^T\| \\ &= \frac{1}{3} \|[1, -2, 1]^T\| = \frac{1}{3} \sqrt{1+4+1} = \frac{1}{3} \sqrt{6} \end{aligned}$$

10. Write

$$\frac{1+i}{2-i} - \frac{3-4i}{1+2i}$$

in the form $a + ib$ with $a, b \in \mathbb{R}$.

- A. $\frac{1}{4}(7 - 8i)$
 B. $-\frac{1}{2}(9 - 5i)$
 C. $\frac{3}{7}(2 - 8i)$
 D. $\frac{4}{5}(6 + i)$
 E. $\frac{1}{5}(6 + 13i)$
 F. $-\frac{2}{3}(5 - 11i)$

My answer: E

$$\begin{aligned} \frac{1+i}{2-i} - \frac{3-4i}{1+2i} &= \frac{(1+i)(2+i)}{4+1} - \frac{(3-4i)(1-2i)}{1+4} = \frac{1}{5} [2+i^2+2i+i - (3+8i^2-4i-6i)] \\ &= \frac{1}{5} (1+3i - (-5-10i)) = \frac{1}{5}(6+13i) \end{aligned}$$

11. The roots of $3x^2 + 2x + 1$ are

- A. $\frac{3}{4}(1 \pm 2i)$
 B. $\frac{1}{3}(-1 \pm \sqrt{2}i)$
 C. This polynomial does not have roots
 D. $\frac{1}{6}(-4 \pm \sqrt{11}i)$
 E. $\frac{2}{3}(1 \pm \sqrt{5})$
 F. $\frac{1}{7}(2 \pm \sqrt{5}i)$

My answer: B

For a quadratic polynomial $ax^2 + bx + c$ the roots are given by $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. We have $a=3$, $b=2$, $c=1$. Hence the roots are

$$\frac{1}{6}(-2 \pm \sqrt{4 - 4 \cdot 3}) = \frac{1}{6}(-2 \pm \sqrt{-8}) = \frac{1}{6}(-2 \pm 2\sqrt{2}i) = \frac{1}{3}(-1 \pm \sqrt{2}i)$$

12. Find the correct combination of true/false for the following three statements.

- For a complex number z and its complex conjugate \bar{z} we always have $z\bar{z} \geq 0$.
- The equation $x^3 - 7x^2 + 43x - 13 = 0$ has a solution in the complex numbers.
- For a complex number z and a real number r we always have $|rz| = |r||z|$.

- (A) true/true/true
 B. false/true/true
 C. true/false/true
 D. true/true/false
 E. false/false/true
 F. false/true/false
 G. false/false/false

all statements are correct.

My answer: A