University of Ottawa
Department of Mathematics and Statistics

MAT 1341D: Introduction to Linear Algebra
Instructor: Catalin Rada

Test 1

Family name (CAPITALS)	______________________
First name (CAPITALS)	______________________
Signature	______________________
Student number	______________________

Please read these instructions carefully:

- The table below is for the TA. Do not write in it.
- For privacy reasons, this page of the assignment will be detached, and you will only get back the remaining pages. Therefore, **fill in your name on both pages and your student number on this page only.**

Question	1	2.	3	4	5	6	7	Total
Score								
Max. score	5	3	2	2	8	8	2 bonus	28

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Please read these instructions carefully:

- Read each question carefully, and answer all questions in the space provided after each question. For questions 5 and 6 you may use the back of pages if necessary, but be sure to indicate to the marker that you have done so.
- No part marks will be given for questions 1 – 4. However, you must show some work to obtain the point. Simply writing the correct answer will earn you 0.
- Question 7 is a bonus proof question. You can get 2 extra points.
- No books or notes are allowed. **Calculators are not permitted.**

Good luck! Bonne chance!

(1) (5 pts) In each case give an example of:
(a) (1 pts) An inconsistent linear system of 2 equations in 3 variables.

Solution: There are many possibilities. For example,

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 = & 0 \\ x_1 & + & x_2 & + & x_3 = & 1 \end{array}$$

(b) (2 pts) A linear system of 2 equations in 2 variables which has a unique solution. Also, give the solution.

Solution: For example,

$$\begin{array}{ccccc} x_1 & + & x_2 & = & 0 \\ & & x_2 & = & 0 \end{array}$$

has the solutions

$$x_1 = 0, \quad x_2 = 0$$

(c) (2 pts) An example of a linear system with infinitely many solutions. Also, give the solutions.

Solution: For example, the linear system $x + y = 1$ has infinitely many solutions: $y = t$, $x = 1 - t$, t scalar.

(2) (3 pts) Complete the theorem below by stating 3 conditions which are equivalent to, but not the same as the condition in (a).

Theorem. *For a $n \times n$ matrix A the following conditions are equivalent:*

(a) *A is invertible.*

(b)

(c)

(d)

Remarque : The theorem stated in class had more equivalent conditions. But you are only asked to list 3 of them.

Solution: Any combination of three of the following is correct: :

- The linear system $AX = B$ has a unique solution for every column B.
- The homogeneous linear system $AX = 0$ has only the trivial solution.
- The reduced row-echelon form of A is the identity matrix I_n.
- A has rank n.
- The linear system $AX = B$ has a solution for every columns B.
- There exists a $n \times n$ matrix C such that $AC = I_n$.
- A^T is invertible.

(3) (2 pts) Consider the following matrices:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & 0 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & -5 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 & 2 & -3 & 5 \\ 0 & 0 & 3 & 0 & -2 \\ 1 & 0 & 1 & -6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Which one is or which ones are in reduced row-echelon form?

My answer:__________________

Solution: B and F; A is not in ref because of columns 3 and 4; C is in ref but not in rref because of column 5; D and E are not in ref because of column 1.

(4) (2 pts) Let $A = \begin{bmatrix} 2 & 7 & 1 \\ 5 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}$. Calculate the matrix $B^2(AB)^{-1}A$.

My answer:__________________

Solution:

$$B^2(AB)^{-1}A = B^2(B^{-1}A^{-1})(A) = (B^2B^{-1})(A^{-1}A) = B\,I_2 = B = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}.$$

(5) For the system of linear equations

$$\begin{array}{rcrcrcr} -x & & -2y & + & 3z & = & -4 \\ 3x & - & y & + & 5z & = & 2 \\ 4x & + & y & + & (a^2-14)z & = & a+2 \end{array}$$

(a) (6 pts) determine the values of a for which the system has

(i) no solution,

(ii) infinitely many solutions,

(iii) a unique solution.

(b) (2 pts) In case (ii) above describe give all solutions.

Solution: The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} -1 & -2 & 3 & -4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right]$$

We perform the following operations, where R_i is row i: $3R_1 + R_2 \to R_2$ and $4R_1 + R_3 \to R_3$; $R_3 - R_2 \to R_2$; $-R_1$, $\frac{-1}{7}R_2$ and obtain:

$$M = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right].$$

Since $a^2 - 16 = (a-4)(a+4)$ we get:

- If $a = -4$, then the last row of M is $\left[\begin{array}{ccc|c} 0 & 0 & 0 & -8 \end{array}\right]$. Hence the system is inconsistent.
- If $a = 4$ alors $M = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Hence the system has infinitely many solutions.
- If $a \notin \{-4, 4\}$, then $M = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & * & * \end{array}\right]$ where the stars "$*$" are non-zero numbers. Hence the system is uniquely solvable, because there does not exist a free variable.

The answer to question (a) is therefore:

(i) The system in inconsistent if $a = -4$.

(ii) The system has infinitely many solutions if $a = 4$.

(iii) The system is uniquely solvable if $a \notin \{4, -4\}$.

To answer (b), let $a = 4$ in the matrix M above. This yields

$$M = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

a matrix in row-echelon form. The leading variables are x, y while z is a free variable. Putting $z = t$ ($t \in \mathbb{R}$) gives the following general solution:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8/7 - t \\ 2t + 10/7 \\ t \end{bmatrix} = \begin{bmatrix} 8/7 \\ 10/7 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

hence the set of solutions is

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8/7 \\ 10/7 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \qquad (t \in \mathbb{R})}$$

(6) (8 pts) In the matrix below **replace α with the second-last digit of your student number** and find its inverse:

$$A = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{bmatrix}$$

Check your answer by verifying $AA^{-1} = I_3$.

Solution: We apply the Inversion Algorithm, i.e., we find the reduced row-echelon form of $[A|I_3]$:

$$\begin{aligned} [A|I_3] &= \left[\begin{array}{ccc|ccc} 1 & \alpha & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & \alpha & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \\ &\sim \left[\begin{array}{ccc|ccc} 1 & \alpha & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 4 & -3 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -4\alpha & 3\alpha \\ 0 & 1 & 0 & 0 & 4 & -3 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \end{aligned}$$

Hence the inverse of A is the 3×3-matrix next to the identity matrix I_3 above:

$$A^{-1} = \begin{bmatrix} 1 & -4\alpha & 3\alpha \\ 0 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix}.$$

We check

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & -5\alpha & 2\alpha \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1+0+0 & -4\alpha+4\alpha & 3\alpha-3\alpha \\ 0 & 0+4-3 & -3+3 \\ 0 & 4-4 & -3+4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

(7) (2 bonus points) (a) Give the definition for a $n \times n$ matrix A to be invertible.
(b) Show that the inverse of an invertible matrix is unique.

Solution: (a) A $n \times n$ matrix A is invertible if there exists a $n \times n$ matrix B such that $AB = I_n = BA$, where I_n is the $n \times n$ identity matrix.

(b) Suppose B and B' satisfy $AB = I_n = BA$ and $AB' = I_n = B'A$. Then $B = BI_n = B(AB') = (BA)B' = I_nB' = B'$.