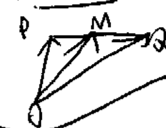
·1148 $\vec{AB} = \begin{pmatrix} -1 \\ 4 \\ 4 \end{pmatrix}$; $\vec{CD} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$. Hence $\vec{AB} \cdot \vec{CD} = 0$.

It follows: $\vec{AB} \perp \vec{CD}$, Done

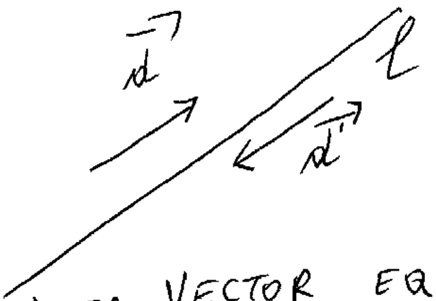
TALK: MIDPOINT !!



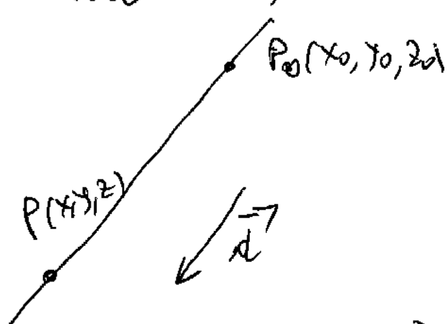
$\vec{OM} = \frac{1}{2}(\vec{OP} + \vec{OQ})$

§ 3.3 LINES and PLANES

LINES Def A direction vector of a line is a NON-ZERO VECTOR that is parallel to the line.



DEF: VECTOR EQ. of a LINE that has $\vec{d}$ as a direction vector and is passing through $P_0(x_0, y_0, z_0)$:



$\vec{d} \parallel \vec{PP_0}$ or $\vec{P_0P}$ so:

$\vec{PP_0} = t\vec{d}$; t scalar

SET $\vec{p} = \vec{OP}$; $\vec{p_0} = \vec{OP_0}$. We get

$\vec{p} - \vec{p_0} = t\vec{d}$ or $\boxed{\vec{p} = \vec{p_0} + t\vec{d}}$

OR: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t\begin{pmatrix} a \\ b \\ c \end{pmatrix}$; where $\vec{d} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Def (Scalar Eq. of a line) (just multiply, identify)

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

Read/Do — 152–153 — ALL examples

DO: 2/158 g, h, c, f

4 a

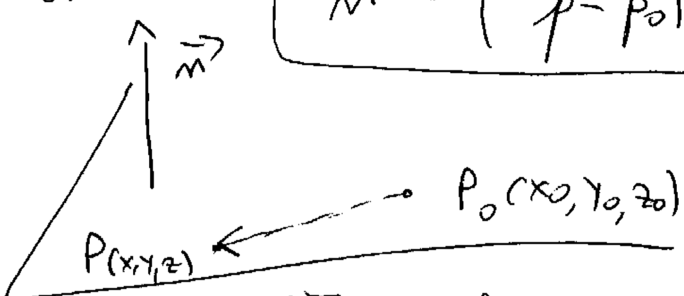
PLANES:

Def: A non-zero $\vec{n}$ is called <u>NORMAL</u> to a plane if it is <u>orthogonal</u> ($\perp$) to every vector in the plane.

$\to$ there are many <u>normals</u> to a given <u>plane</u>!!

DEF (<u>VECTOR EQ. of a PLANE</u>)
Consider the plane with normal $\vec{n} \neq \vec{0}$ which contains the point $P_0(x_0, y_0, z_0)$. <u>THEN</u> a point $P(x,y,z)$ lies in this plane $\iff$ $\vec{n} \cdot \overrightarrow{PP_0} = 0$

OR: $\boxed{\vec{n} \cdot (\vec{p} - \vec{p_0}) = 0}$, where $\vec{p} = \overrightarrow{OP}$; $\vec{p_0} = \overrightarrow{OP_0}$.



if $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ then

Def (<u>Scalar Eq. of the plane</u>) $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

<u>OR:</u> $ax + by + cz = ax_0 + by_0 + cz_0$

<u>Do:</u> $\boxed{9/158}$

<u>SOL:</u>
- $2 \cdot 1 - 3(-2) + (-2) = 6 \Rightarrow$ P is in this plane
- $2 \cdot (5) - 3(-6) + 0 = 10 + 18 \neq 6 \Rightarrow$ Q is NOT !!

<u>THM</u> Let $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \vec{0}$.

(1°) Every plane with normal $\vec{n}$ has the eq: $ax + by + cz = K$, for some K.

(2°) For every K, the graph of eq. $ax + by + cz = K$ is a plane with normal $\vec{n}$.

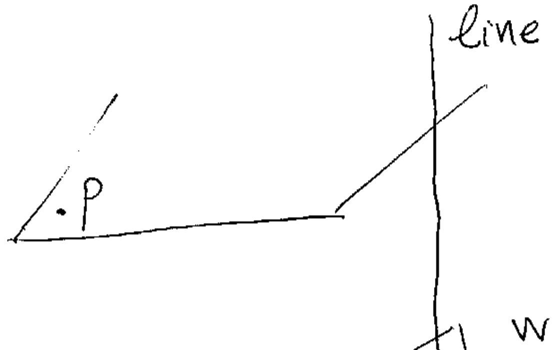
$\boxed{\text{DO: } 10b/159}$ SOL: The eq. is $x-2y+3z=k$, for some k (since the planes are parallel).

Plug in $P(1,-2,4)$ and get k: $k = 1-2(-2)+3\cdot 4 = 17$

<u>Hence</u>: $x-2y+3z=17$

$\boxed{\text{DO: } 10c/159}$ Contains $P(2,-3,0)$; $\perp$ to the line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} + t\begin{pmatrix} 6 \\ -6 \\ 5 \end{pmatrix}$.

<u>SOL</u>:

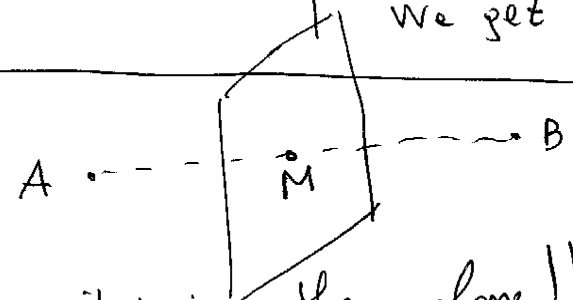


$\vec{n} = \vec{d} = \begin{pmatrix} 6 \\ -6 \\ 5 \end{pmatrix}$

<u>so</u>: $\begin{pmatrix} 6 \\ -6 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} x-2 \\ y-(-3) \\ z-0 \end{pmatrix} = 0$

We get $6x-6y+5z=30$

$\boxed{\text{DO: } 10f/159}$



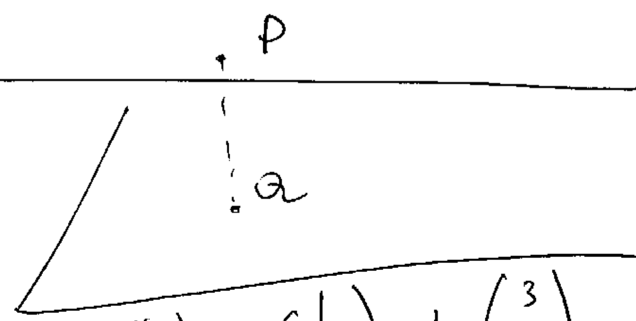
M = midpoint

$M\left(\frac{7}{2}, \frac{1}{2}, -\frac{1}{2}\right)$ it is in the plane!! How we get a normal

$\vec{n} = \vec{AB} = \begin{pmatrix} -5 \\ 1 \\ -5 \end{pmatrix}$. <u>Hence</u>: $\begin{pmatrix} -5 \\ 1 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} x-7/2 \\ y-1/2 \\ z+\frac{1}{2} \end{pmatrix} = 0 \Rightarrow$

$$\boxed{-5x+y-5z = -\frac{29}{2}}$$

$\boxed{\text{DO: } 23a/159 \quad \text{TRY HOME b)}}$



SOL: <u>NOTE</u> $\vec{PQ} \perp$ plane

<u>Since</u> $\vec{n} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} \Rightarrow$ the line passing through P, Q is: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$.

Since Q is on the line $\Rightarrow Q(1+3t, -t, -2+4t)$ for some scalar t. BUT Q is in the plane:

$3(1+3t)-(-t)+4(-2+4t)-5 \Rightarrow t = \frac{10}{26} = \frac{5}{13}$.

Hence $Q\left(\frac{28}{13}, -\frac{5}{13}, -\frac{6}{13}\right)$. The distance is:

$$\|PQ\| = \sqrt{\left(\frac{28}{13}-1\right)^2 + \left(-\frac{5}{13}-0\right)^2 + \left(-\frac{6}{13}+2\right)^2} = \text{----}$$

CROSS PRODUCT:

Q: HOW can we find a non-zero vector $\perp$ on 2 given non zero vectors?

A: cross product

Def Let $\vec{v} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$. Then

$$\vec{v} \times \vec{w} = \begin{bmatrix} y_1 z_2 - y_2 z_1 \\ -(x_1 z_2 - x_2 z_1) \\ x_1 y_2 - x_2 y_1 \end{bmatrix}$$ is called the cross product of $\vec{v}$ and $\vec{w}$.

THM
$\begin{cases} 1^\circ \quad \vec{v} \cdot (\vec{v} \times \vec{w}) = 0 = \vec{w} \cdot (\vec{v} \times \vec{w}), \text{ So } \vec{v}, \vec{w} \perp \vec{v} \times \vec{w} \\ 2^\circ \quad \vec{v} \times \vec{w} = \vec{0} \iff \vec{v}, \vec{w} \text{ are parallel.} \end{cases}$

DO 13/159 e) $A(3,1,2)$, $B(5,-1,3)$, $C(-4,2,0)$

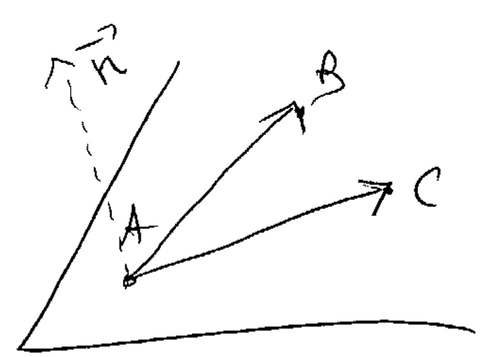
SOL: $\vec{AB}$, $\vec{AC}$ are in the plane !!!

We need a normal: $\vec{n} = \vec{AB} \times \vec{AC} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} -7 \\ 1 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ -(3) \\ -12 \end{pmatrix}$. Hence the eq. is:

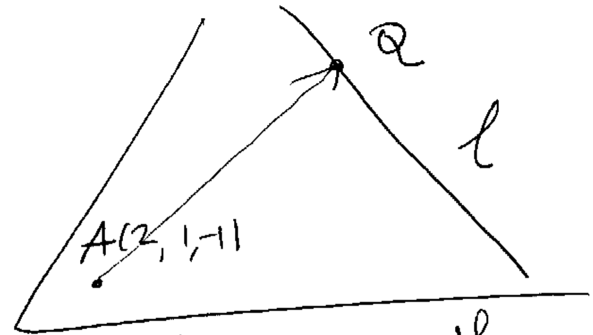
$$\begin{pmatrix} 3 \\ -3 \\ -12 \end{pmatrix} \cdot \begin{pmatrix} x-3 \\ y-1 \\ z-2 \end{pmatrix} = 0$$ (since A is in the plane).

SO: $x - y + 4z = 10$



$\boxed{\text{DO 13 } g/159}$ Containing $A(2,1,-1)$ and the line

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}.$$



<u>SOL:</u> $t=0 \Rightarrow Q\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ is in the plane,

so $\overrightarrow{AQ}$ is in the plane:

$$\overrightarrow{AQ} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

$$\vec{n} = \overrightarrow{AQ} \times \vec{d} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix}.$$ Hence the

eq. <u>is</u> $\begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x-2 \\ y-1 \\ z+1 \end{pmatrix} = 0 \Rightarrow \boxed{-3x - 2y - z = -7}$

$\boxed{\text{MORE on DGD}}$

§ 3.5 CROSS PRODUCT

Th (Algebraic properties of cross product)

1° $\vec{v} \times \vec{w}$ is a vector; 2° $\vec{v} \times \vec{0} = \vec{0} = \vec{0} \times \vec{v}$

3° $\vec{v} \times \vec{v} = \vec{0}$ 4° $\vec{w} \times \vec{v} = -(\vec{v} \times \vec{w})$

5° $(a\vec{v}) \times \vec{w} = a(\vec{v} \times \vec{w}) = \vec{v} \times (a\vec{w})$; a #

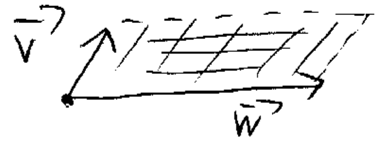
6° $\vec{v} \times (\vec{u} + \vec{w}) = \vec{v} \times \vec{u} + \vec{v} \times \vec{w}$

7° $(\vec{u} + \vec{w}) \times \vec{v} = \vec{u} \times \vec{v} + \vec{w} \times \vec{v}$

TH $\|\vec{v} \times \vec{w}\|^2 = \|\vec{v}\|^2 \cdot \|w\|^2 - (\vec{v} \cdot \vec{w})^2.$

TH If θ is the angle between the vectors: $\vec{v}, \vec{w}$, then:

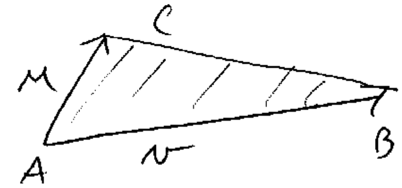
$\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \cdot \|\vec{w}\| \sin\theta$ = Area of Parallelogram determined by $\vec{v}, \vec{w}$



DO 1e/179

Area of triangle ABC.

$A(1,-1,2)$; $B(3,3,2)$; $C(5,0,-4)$



SOL: $A = \frac{1}{2} \| \vec{u} \times \vec{v} \| =$

$$= \frac{1}{2} \left\| \begin{pmatrix} 4 \\ 1 \\ -6 \end{pmatrix} \times \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} \right\|$$

$$= \frac{1}{2} \left\| \begin{pmatrix} 24 \\ -12 \\ 14 \end{pmatrix} \right\| = \frac{1}{2} \sqrt{(24)^2 + (-12)^2 + (14)^2} = \sqrt{229}$$

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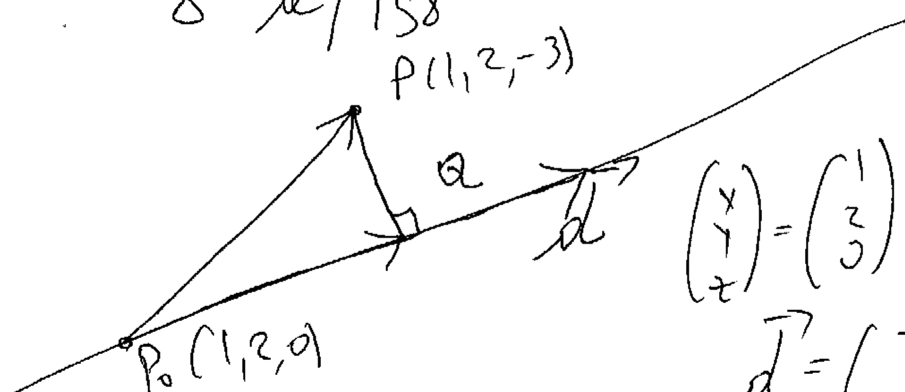
if TIME : 8 e/158



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix}$$

$\vec{d} = \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix}$ direction vector

$$\overrightarrow{P_0Q} = \operatorname{proj}_{\vec{d}}(\overrightarrow{P_0P}) =$$

$$= \frac{\overrightarrow{P_0P} \cdot \vec{d}}{\|\vec{d}\|^2} \cdot \vec{d} = \frac{\begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix}}{50} \vec{d} = \frac{-12}{50} \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix}$$

$Q(\alpha, \beta, \gamma) \Rightarrow$

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \frac{-12}{50} \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{86}{50} \\ \frac{40}{50} \\ \frac{-48}{50} \end{pmatrix}$$

$$\|PQ\| = \sqrt{\left(\frac{86}{50} - 1\right)^2 + \left(\frac{40}{50} - 2\right)^2 + \left(-\frac{48}{50} + 3\right)^2} = \cdots$$

SOLUTIONS OF 3 EXCS:

21 a / 3.3 / pg 159 Let Q be the intersection point. Since Q is on the line $\Rightarrow$
$Q(2+4t, -t, -1+2t)$ for some t.
But Q is also in the plane:

$$2+4t-3(-t)+2(-1+2t)=7$$
$$\Rightarrow \quad t=\frac{7}{11}$$

So $Q\left(2+\frac{4\cdot 7}{11}, -\frac{7}{11}, -1+2\cdot\frac{7}{11}\right)$; i.e.
$Q\left(\frac{50}{11}, \frac{-7}{11}, \frac{3}{11}\right)$

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20 a / 159 Every point Q on the line has the form: $Q(3+2t, 2, 1-4t)$ for some # t.
Since $2(3+2t)+(-3)\cdot 2+(1-4t)=6+4t-6$
$+1-4t=1 \Rightarrow Q$ is in the plane.
Since Q is arbitrary taken on the line, the whole line is in the plane

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13 j / 159

1st line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + t \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$

$\vec{d_1}$ direction vector

2nd line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ 7 \end{pmatrix} + t \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$

$\vec{d_2}$ direction vector

We need a normal.
Since $\vec{d_1}, \vec{d_2}$ are in the plane (and NOT parallel) we get $\vec{n} = \vec{d_1} \times \vec{d_2} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \times \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} 8 \\ -3 \\ -6 \end{pmatrix}$. Hence the eq. is

$\vec{n} \cdot \begin{pmatrix} x-1 \\ y-1 \\ z+2 \end{pmatrix} = 0$

for $t=0$ in the 1st line we get $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ is in the plane

Hence
$8x - 8 + (-3)(y-1) + (-6)(z+2) = 0$

$$\boxed{8x - 3y - 6z = 17}$$