Lecture 9 - TUE - FEB - 3 - 2009

(1.5)

Def A square mx. is called <u>Upper Triangular</u> if every entry <u>below</u> the <u>main</u> diagonal is zero. A square mx. is called <u>LOWER Triangular</u> if every entry <u>ABOVE</u> the main diagonal is zero.

<u>EXP</u>: $\begin{pmatrix} 3 & -1 & 2 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; $\begin{pmatrix} -3 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

THM Let A be a <u>triangular</u> mx. (Upper or Lower).

(1) A is invertible <=> no entry on the main diagonal is 0.

(2) If A is Upper (Lower) triangular, then A^{-1} is Upper (Lower) triangular.

GIVE IDEA of proof! <u>EXP</u>: $\begin{pmatrix} 3 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & \boxed{0} \end{pmatrix}$ IT IS NOT <u>invertible</u>

§§ CONDITION <u>for INVERTIBILITY</u>

THM The following conditions are equivalent for a square mx:

1. A is invertible
2. $AX = 0$ has <u>ONLY</u> the <u>trivial Solution</u>.
3. $AX = B$ <u>has</u> a solution for <u>each choice</u> of B
4. A can be transformed into I_n by Row Operations
5. There is a mx. C such that $AC = I_n$.

DO: 19/48 (a), 14/48, 16/48, 13/48

Chapter 2: Determinants AND EIGENVALUES

2.1 COFACTOR expansions

Remember: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is inv. $\iff$ $ad-bc \neq 0$.

Call $ad-bc = \det(A)$. So: if $\det(A) \neq 0 \Rightarrow A$ inv.
Also $\Leftarrow$ WORKS. What if A is 1×1? SAME answer: A inv
$\iff A \neq 0$. What if A is 3×3, 20×20???

$\Rightarrow$ Suppose we know how to compute determinants of $(n-1)\times(n-1)$ matrices.

$\to$ Let A be an $n \times n$ mx.

$\to$ Let A_{ij} be the $(n-1)\times(n-1)$ mx obtained from A by deleting Row i and Column j.

$\to$ DEFINE: the (i,j)-cofactor $C_{ij}(A)$ of A by

$$C_{ij}(A) = (-1)^{i+j} \det(A_{ij}) \quad \text{for each } i, j.$$

EX: DO 1a /80.

$\to$ Let us say $A = [a_{ij}]$.

$\to$ DEF the determinant of A, $\det A$, is given by

$$\det A = a_{11} C_{11}(A) + a_{12} \cdot C_{12}(A) + \dots + a_{1n} C_{1n}(A).$$

(i.e., multiply each entry in $\boxed{\text{Row 1}}$ of A by the corresponding cofactor, then add the results)

$\to$ NOTE: We get back the case 2×2!!! ☺

DO: EX: det 3f / 80 !!

→ The expression $\det A = a_{11} C_{11}(A) + a_{12} C_{12}(A) + \dots + a_{1n} C_{1n}(A)$ is called THE COFACTOR expansion along Row 1.

→ More generaly: if A is square, the cofactor expansion along any Row (or Column) is a # defined as follows: Multiply each entry in the Row (or Column) by the corresponding cofactor, then ADD the results.

$\boxed{\text{THM}}$ (Cofactor Expansion Theorem) if A is a square mx, the determinant $\det A$ is equal to the expansion along ANY Row or Column of A

$\boxed{\text{DO } 3\,h/80}$

COR: if a mx A has a Row or Column of zeros, then $\det A = 0$.

---

§ ELEMENTARY ROW Operations AND determinants

THM) Let A be an $n \times n$ mx.

(1) if B is obtained from A by Interchanging 2 different Rows/C, then $\det B = -\det A$.

(2) if B is obtained from A by multiplying a Row/Column by a #k, then $\det B = k \cdot \det A$

(3) if B is obtained from A, by adding a multiple of some Row/Column to a different Row/Column then $\det B = \det A$

↳ COR: If a mx. A has 2 identical Rows/Columns, then $\det(A) = 0$

DO: 13/81 ~~ok~~ (using Row op) step
20/81

(THM) $\boxed{\text{If } A \text{ is an } n\times n \text{ mx}, \Rightarrow \det(kA) = k^n \det A, \ \forall k}$

(THM) if a square mx. A is triangular, then $\det A$ is the product of the entries on the main diagonal.

(THM) If A is a square mx, then $\det A = \det A^T$

8/81, 15, 18 a/81, 19

Lecture - 10 - THU - Feb. 5-2009

EXC: 20/81

SOL: $\det \begin{pmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{pmatrix} = ?$ Use Row Operations:

$$\underset{A}{\begin{pmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{pmatrix}} \xrightarrow[R_3-R_1]{R_2-R_1} \underset{B}{\begin{pmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{pmatrix}}$$

$\det B = \det A$; But $\det B = 1\cdot(-1)^{1+1} \det \begin{pmatrix} y-x & y^2-x^2 \\ z-x & z^2-x^2 \end{pmatrix}$

$= (y-x)(z-x)(z+x) - (z-x)(y-x)(y+x) = (y-x)(z-x)(z+x-y-x)$

$= (y-x)(z-x)(z-y)$. Done

2.1 THM If A is an $n\times n$ mx., then $\det(kA) = k^n \det A$ for every #k.

THM If a square mx. A is triangular, then $\det A$ is the product of the entries on the main diagonal.

THM If A is a square mx, then $\det A = \det A^T$.

Do: 19/81

§ 2.2 Determinants AND Inverses

THM(PRODUCT Theorem) if A and B are $n\times n$, $\det(AB) = \det(A)\det B$

COR: ① $\det(A^k) = (\det(A))^k$; ② $\det(A_1 A_2 \cdots A_k) = (\det A_1)\cdots\cdots(\det A_k)$

(Induction . . . -)-)- -)

THM Let A be a square mx. Then:

① A is invertible $\iff$ $\det A \neq 0$

② If A is invertible $\Rightarrow$ $\det(A^{-1}) = \dfrac{1}{\det(A)}$

DO: 4g/88 $A^{-1} = A^{T} \Rightarrow \det(A^{-1}) = \det(A^{T}) \Rightarrow \frac{1}{\det A} = \det A$

$\Rightarrow (\det A)^2 = 1 \Rightarrow \boxed{\det A = \pm 1}$

4e/88 $A^2 + I = 0 \Rightarrow \det(A^2) = \det(-I) \Rightarrow$

$(\det(A))^2 = (-1)^n \Rightarrow \boxed{\det A = \pm i}$ if n = ODD.

THM Let A, B be square matrices. Then

$$\det\begin{pmatrix} A & X \\ 0 & B \end{pmatrix} = (\det A)(\det B) = \det\begin{pmatrix} A & 0 \\ X & B \end{pmatrix}$$

Pf:
$$\begin{pmatrix} A & X \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \cdot \begin{pmatrix} A & X \\ 0 & I \end{pmatrix}$$

$$\det\begin{pmatrix} A & X \\ 0 & B \end{pmatrix} = \det\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \det\begin{pmatrix} A & X \\ 0 & I \end{pmatrix}$$

$$= \det B \cdot \det(A)$$ Done

(expansions across Row 1) (expansion across LAST Row)

Do: 16/a/89

SOL: $$\det\begin{pmatrix} A & X & Y \\ 0 & B & Z \\ 0 & 0 & C \end{pmatrix} \overset{\text{Thm Above}}{=} \det\begin{pmatrix} A & X \\ 0 & B \end{pmatrix} \det C =$$

Square

Square

$$= (\det A)(\det B)(\det C) =$$

$$= 3 \cdot (-1) \cdot 2 = -6$$

ONE more: 7/88 $3\times 3 \Rightarrow 2^3 \det A^{-1} = 5 \Rightarrow$

$$\frac{2^3}{5} = \frac{1}{\det A^{-1}} = \frac{1}{\frac{1}{\det A}} \Rightarrow \boxed{\det A = \frac{2^3}{5}}$$

next: $5 = \det\left[A^2 (B^T)^{-1}\right] = (\det A)^2 \det\left((B^T)^{-1}\right) =$

$= (\det A)^2 \cdot \dfrac{1}{\det B^T} \Rightarrow \det B^T = \dfrac{(\det A)^2}{5} \Rightarrow$

$\det B = \dfrac{(2^3/5)^2}{5} = \dfrac{(2^3)^2}{5^3} = \dfrac{2^6}{5^3}$

---

§§ ADJOINT of a MATRIX

Def Let $A = [a_{ij}]$ be an $n \times n$ mx. The ADJOINT of A is the mx: $\operatorname{adj} A = \left[C_{ij}(A)\right]^T$ (the transpose of the mx. of cofactors)

THM Let A be a square mx.

① (Adjoint Formula) $A \cdot (\operatorname{adj} A) = (\det A)\, I = (\operatorname{adj} A) \cdot A$

② if $\det A \neq 0 \Rightarrow$ $\boxed{A^{-1} = \dfrac{\operatorname{adj} A}{\det A}}$

DO 8b/88:

$\det A = (-2)\cdot(-1)^{1+1} \det\begin{pmatrix} -3 & 1 \\ 5 & 1 \end{pmatrix} - (2)\det\begin{pmatrix} 3 & 1 \\ 5 & 1 \end{pmatrix} =$

$= (-2)(-3-5) - 2(3-5) = (-2)(-8) - 2(-2) = 16 + 4 = 20 \neq 0$

$C_{11}(A) = (-1)^{1+1} \det\begin{pmatrix} -3 & 1 \\ 5 & 1 \end{pmatrix} = -3-5 = -8$

$C_{12}(A) = (-1)^{1+2} \det\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = -(2-0) = -2$

$C_{13}(A) = (-1)^{1+3} \det\begin{pmatrix} 2 & -3 \\ 0 & 5 \end{pmatrix} = 10$

$C_{21}(A) = (-1)^{2+1} \det\begin{pmatrix} 3 & 1 \\ 5 & 1 \end{pmatrix} = -(3-5) = 2$

$C_{22}(A) = (-1)^{2+2} \det\begin{pmatrix} -2 & 1 \\ 0 & 1 \end{pmatrix} = -2$

$C_{23}(A) = (-1)^{2+3} \det\begin{pmatrix} -2 & 3 \\ 0 & 5 \end{pmatrix} = -(-10) = 10$

$C_{31}(A) = (-1)^{3+1} \det\begin{pmatrix} 3 & 1 \\ -3 & 1 \end{pmatrix} = 3+3 = 6$

$C_{32}(A) = (-1)^{3+2} \det\begin{pmatrix} -2 & 1 \\ 2 & 1 \end{pmatrix} = -(-2-2) = 4$

$C_{33}(A) = (-1)^{3+3} \det\begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix} = 0$

Hence $A^{-1} = \frac{1}{20}\begin{bmatrix} -8 & -2 & 10 \\ 2 & -2 & 10 \\ 6 & 4 & 0 \end{bmatrix}^T = \frac{1}{20}\begin{bmatrix} -8 & 2 & 6 \\ -2 & -2 & 4 \\ 10 & 10 & 0 \end{bmatrix}$

---

§§ CRAMER's Rule: Consider the system $AX = B$, A invertible $(n \times n)$; $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. We define $A_i(B)$ the $n \times n$ mx. obtained from A by replacing column i by B

THM For each $i = 1, 2, \dots, n$ one has

$$x_i = \frac{\det(A_i(B))}{\det(A)}$$

DO: 15a/89 $A = \begin{pmatrix} 2 & -5 & 7 \\ -1 & 4 & 2 \\ 3 & 3 & -6 \end{pmatrix}$; $B = \begin{pmatrix} 9 \\ -2 \\ 5 \end{pmatrix}$

So: $\det(A) = 2\det\begin{pmatrix} 4 & 2 \\ 3 & -6 \end{pmatrix} + (-5)(-1)\det\begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix} + 7\det\begin{pmatrix} -1 & 4 \\ 3 & 3 \end{pmatrix}$

$= 2(-24 - 6) + 5(6 - 6) + 7(-3 - 12) = -60 + 7(-15)$

$= -60 - 105 = -165$

$A_1(B) = \begin{pmatrix} 9 & -5 & 7 \\ -2 & 4 & 2 \\ 5 & 3 & -6 \end{pmatrix} = 9\det\begin{pmatrix} 4 & 2 \\ 3 & -6 \end{pmatrix} - (-5)\det\begin{pmatrix} -2 & 2 \\ 5 & -6 \end{pmatrix} +$

$+ 7\det\begin{pmatrix} -2 & 4 \\ 5 & 3 \end{pmatrix} = 9(-24 - 6) + 5(12 - 10) + 7(-6 - 20)$

$= -270 + 10 \cancel{+182} = -270 - 172 = -442$

So $x_1 = \frac{-442}{-165}$

etc.

IF TIME: 21/89: $\det\left(-A^2 (\operatorname{adj} A)^{-1}\right) = (-1)^3 (\det A)^2 \det\left((\operatorname{adj} A)^{-1}\right)$

$= (-1)^3 (\det A)^2 \det\left(A \quad \right) =$

$$= (-1)^3 (\det A)^2 \frac{1}{(\det A)^3} \det A = (-1)^3 = -1$$

12/88, 13/88, 22/89

(2.2) §§ CRAMER's RULE Consider the system $AX = B$, A invertible ($n \times n$); $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. We define $A_i(B)$ the $n \times n$ mx. obtained from A by replacing column i by B.

THM For each $i = 1, 2, \dots, n$ one has: $x_i = \dfrac{\det(A_i(B))}{\det(A)}$

Do 15a/89 $A = \begin{pmatrix} 2 & -5 & 7 \\ -1 & 4 & 2 \\ 3 & 3 & -6 \end{pmatrix}$; $B = \begin{pmatrix} 9 \\ -2 \\ 5 \end{pmatrix}$

SOL: $\det A = 2 \cdot \det \begin{pmatrix} 4 & 2 \\ 3 & -6 \end{pmatrix} + (-5)(-1) \det \begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix} + 7 \cdot \det \begin{pmatrix} -1 & 4 \\ 3 & 3 \end{pmatrix} = 2(-24-6) + 7(-3-12) = -60 - 105 = -165$

$A_1(B) = \begin{pmatrix} 9 & -5 & 7 \\ -2 & 4 & 2 \\ 5 & 3 & -6 \end{pmatrix} \to \det A_1(B) = 9 \det \begin{pmatrix} 4 & 2 \\ 3 & -6 \end{pmatrix} + (-5)(-1) \cdot \det \begin{pmatrix} -2 & 2 \\ 5 & -6 \end{pmatrix} + 7 \det \begin{pmatrix} -2 & 4 \\ 5 & 3 \end{pmatrix} = 9(-24-6) + 5(12-10) + 7(-6-20) = 9 \times (-30) + 10 - 182 = -270 - 172 = -442$.

Hence $\boxed{x_1 = -\dfrac{442}{-165}}$ etc

21/89: $\det(-A^2 (\operatorname{adj} A)^{-1}) = (-1)^3 \det(A^2) \det((\operatorname{adj} A)^{-1}) = -(\det A)^2 \det\left(\dfrac{A}{\det A}\right) = -(\det A)^2 \cdot \dfrac{1}{(\det A)^3} \det A = \boxed{-1}$.

12/88, 13/88; 22/89 → EASY

JUMP

CHAPTER 4 The VECTOR Space $\mathbb{R}^n$

Def If $n \geq 1$, an ordered n-tuple: $(a_1, a_2, \ldots, a_n)$ of n real #s is called n-vector. The set of all n-vectors is called n-space. Notation: $\mathbb{R}^n$.

Def Two n-vectors are equal $\iff$ the corresponding entries are equal.

$\mathbb{R}^1$ is just $\mathbb{R}$ – the set of all real #s.

Def A set U of vectors in $\mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if:

1. The zero vector $0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ is in U
2. if X, Y are in U, then X + Y is in U
3. if X is in U, r is a scalar then rX is in U.

(closed under +, ·)

EXP: • $\mathbb{R}^n$; $\{0\}$ ← the zero subspace of $\mathbb{R}^n$.

• A subspace of $\mathbb{R}^n$ other than $\{0\}$, $\mathbb{R}^n$ is called a proper subspace of $\mathbb{R}^n$.

Read EXP 5/183 lines through origin, planes through origin > Are subspaces of $\mathbb{R}^3$.

5d/188 → NO: The zero vector is not in

More examples (from the world of matrices)

Let A be an m × n mx. A. The null space of A, null $A = \{X \in \mathbb{R}^n \mid AX = 0\}$. The image of A, $\text{Im } A = \{Y \text{ in } \mathbb{R}^m \mid Y = AX \text{ for some } X \text{ in } \mathbb{R}^n\}$.

IMPT: $\begin{cases} \text{null } A \subseteq \mathbb{R}^n; \\ \text{Im } A \subseteq \mathbb{R}^m. \end{cases}$

(THM) • null A is a subspace of $\mathbb{R}^n$.
• im A is a subspace of $\mathbb{R}^m$.

Pf: * Y_1, Y_2 in im A $\to$ $Y_1 = AX_1$, $Y_2 = AX_2$ for some X_1, X_2 in $\mathbb{R}^n$! Then $Y_1 + Y_2 = AX_1 + AX_2 = A(X_1 + X_2)$.

So $Y_1 + Y_2$ is in im A.

etc

More Examples: (1) Given vectors $X_1, X_2, \dots, X_k$ in $\mathbb{R}^n$, the set of all linear combinations of the X_i is called the span of $X_1, X_2, \dots, X_k$.

Notation: $\text{span}\{X_1, X_2, \dots, X_k\} = \{t_1X_1 + \dots + t_kX_k \mid t_i \text{ in } \mathbb{R}\}$

(THM) Let $X_1, X_2, \dots, X_k$ be any vectors in $\mathbb{R}^n$.

(1) $\text{span}\{X_1, X_2, \dots, X_k\}$ is a subspace of $\mathbb{R}^n$ which contains each of the vectors $X_1, X_2, \dots, X_k$.

(2) If $X_1, \dots, X_k$ all lie in some subspace V, then
$\text{span}\{X_1, X_2, \dots, X_k\} \subseteq V$.

Pf: (1). $0 = 0X_1 + \dots + 0 \cdot X_k$; • If $X = t_1X_1 + \dots + t_kX_k$, $Y = s_1X_1 + \dots + s_kX_k$ $\Rightarrow$ $X + Y = (t_1 + s_1)X_1 + \dots + (t_k + s_k)X_k$

• If $X = t_1X_1 + \dots + t_kX_k$, $c\#$ $\Rightarrow$ $cX = (ct_1)X_1 + \dots + (ct_k)X_k$

(2) easy...

~~DO:~~

4c/187

DO: 10/188; 13/188

Def: If a subspace U has the form: $U = \text{span}\{X_1, \dots, X_k\}$ we say that $X_1, \dots, X_k$ are a spanning set of U (or U is spanned by the X_i's).

Exp: $\mathbb{R}^n = \text{span}\{E_1, E_2, \ldots, E_n\}$, where $E_1, E_2, \ldots, E_n$ are the columns of I_n.

EASY

VERY IMPT:

EXP: Let A be an $m\times n$ mx, let $\{X_1, \ldots, X_k\}$ be the basic solutions of $AX=0$. Then

null $A = \text{span}\{X_1, X_2, \ldots, X_k\}$

→ Recall: §1.3

EXP: if $A = [C_1 \; C_2 \; \cdots \; C_n]$, then $\text{im}\, A = \text{Span}\{C_1, C_2, \ldots, C_n\}$.

SOL: $AX = [C_1 \; C_2 \; \cdots \; C_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{x_1 C_1 + \cdots + x_n C_n}_{\text{l.combo}}$

15a/189 18 → easy $U = \text{null}(A-B)$ → ITIS ...

14/189 ; 8d/188 ; 24a/188
c