DEF: if $X_1, X_2, \dots, X_k$ are columns, the expression $s_1 X_1 + s_2 X_2 + \dots + s_k X_k$, where $s_1, s_2, \dots, s_k$ are #s is called a linear combination of $X_1, X_2, \dots, X_k$.

THM if a homogeneous system in n variables has the augmented mx. with rank r, then:

1° There are $\boxed{n - r}$ basic [redacted] SOLUTIONS.

2° Every solution is a linear combination of the basic solutions.

DO: 8/26; 7c/26 (t); 4a/25, b, c, 3≤ ≤6

§ 1.4 MATRIX MULTIPLICATION

Def: Let $R = [r_1 \ r_2 \ \dots \ r_n]$ be a Row mx;
Let $C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ be a Column mx. We define their DOT PRODUCT to be the #: $\boxed{r_1 c_1 + r_2 c_2 + \dots + r_n c_n}$

EXP: $[2 \ 0 \ 1] \cdot \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = 2(-1) + 0 \cdot 1 + 1(3) = 1$

DEF: The product of an $m \times n$ mx. A and an $n \times p$ mx B is the $m \times p$ mx AB whose (i,j)-entry is the DOT PRODUCT of ROW i of A AND COLUMN j of B.

DO: 1f/36 $\begin{pmatrix} 4 & 2 \\ -2 & -5 \\ 1 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & 5 \end{pmatrix} = \dots\dots$

(3×2) (2×3) (3×3)

Rule: A times B = AB

$m \times n$ $n \times p$ $m \times p$

the same

We say A B are compatible for multiplication

<u>DEF:</u> If A is a square mx., the powers of A are:
$A^2 = A\cdot A$; $A^3 = A\cdot A\cdot A$, etc.

$\boxed{\text{FACT}}$ $\begin{cases} A^2 = 0 \not\Rightarrow A = 0 \text{ ; EXP: } A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ AB \neq BA \text{ ; EXP: } A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ; B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \end{cases}$

<u>Def:</u> I_n is the $n\times n$ mx. with 1's on the main diagonal and zeros elsewhere. (Plays the Role of 1!!!)

<u>ALGEBRAIC PROPERTIES</u> Let c #, A, B, C be matrices with sizes such that the products below can be performed:
<u>Then:</u> 1°) $IA = A$; $BI = B$ (2) $A(BC) = (AB)C$; (3) $A(B \pm C) = AB \pm AC$; (4) $(B \pm C)A = BA \pm CA$; (5) $c(AB) = (cA)B = A(cB)$; (6) $(AB)^T = B^T \cdot A^T$

<u>EXC:</u> If $AB = I$ and $CA = I$, then $B = C$

<u>SOL:</u> $AB = I \Rightarrow C(AB) = CI \Rightarrow (CA)B = C \Rightarrow IB = C \Rightarrow B = C$

---

§§ MATRIX MULTIPLICATION And LINEAR SYSTEMS

<u>EXP:</u> Consider the system: $\begin{cases} x_1 - 2x_2 + 3x_3 + 4x_4 = 5 \\ 3x_1 + 6x_2 + 7x_3 - x_4 = 0 \end{cases}$.

We get: $\begin{pmatrix} x_1 - 2x_2 + 3x_3 + 4x_4 \\ 3x_1 + 6x_2 + 7x_3 - x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -2 & 3 & 4 \\ 3 & 6 & 7 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$,

or $AX = B$, where $A = \begin{pmatrix} 1 & -2 & 3 & 4 \\ 3 & 6 & 7 & -1 \end{pmatrix}$; $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$; $B = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$.

They are called: <u>coefficient mx</u>; <u>mx. of variables</u> and <u>mx. of constants</u>, respectively.

$\Rightarrow$ Any linear system can be written in this <u>form</u>.

→ Given a system $AX=B$, we may form a new system: $AX=0$; it is called the associated homogeneous system. Their solution sets are related:

THM Suppose X_0 is a particular solution of $AX=B$.

(1) If X' is a solution of $AX=0$, then $X=X_0+X'$ is a solution of $AX=B$.

(2) Every solution X of $AX=B$ has the form: $X=X_0+X'$ for some X' solution of $AX=0$

Pf: Easy exercise

DO 13/36

§ $\boxed{\text{BLOCK multiplication}}$

THM Let $A=[C_1 \ C_2 \ \cdots \ C_n]$ be an $m\times n$ mx. with columns $C_1, C_2, \ldots, C_n$.

(1) If B is a $k\times m$ mx, then $BA = B[C_1 \ C_2 \ \cdots \ C_n] \overset{!}{=}$

$\overset{!}{=} [BC_1 \ BC_2 \ \cdots\cdots \ BC_n]$

(2) If $X=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a column, then $AX = AX = [C_1 \ C_2 \ \cdots \ C_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$= x_1C_1 + x_2C_2 + \cdots + x_nC_n$

EXP: 1) DO: $\underset{B}{\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}}\underset{A}{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}} = \left[B\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad B\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]$ and DO:

$BA = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$. Compare the results.

2) DO: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = 5\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 6\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \cdots\cdot$

and DO: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}$ and compare the results.

MORE IS TRUE: say: $A = \left[\begin{array}{c|c} P & X \\ \hline 0 & Q \end{array}\right]$; $B = \begin{bmatrix} U \\ V \end{bmatrix}$; then $AB = \begin{bmatrix} PU + XV \\ QV \end{bmatrix}$,

if the blocks are compatible for multiplication.

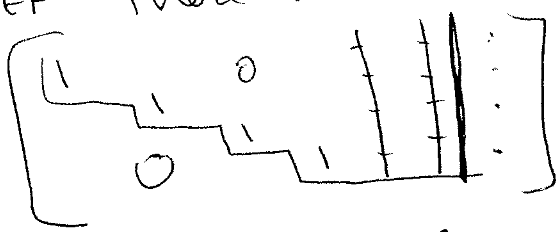
DO on exp: 23/37.

DO 25a/ $\begin{pmatrix} 1 & X \\ -Y & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix} = \begin{pmatrix} 1+XY & X \\ 0 & 1 \end{pmatrix}$

31/37 EASY

16/37 4×6, rank is 4 $\Rightarrow$ YES:

In RREF there is a leading 1 in every Row:



17/37 $AX = 0 \Rightarrow BAX = B0 \Rightarrow 1X = 0 \Rightarrow X = 0$,

contradiction

15/37 RREF of our setting is $\left[\begin{array}{ccc|c} 1 & & 0 & b \dots \\ & 1 & & b \dots \\ 0 & & 1 & b \dots \end{array}\right] \Rightarrow$

unique solution.

---

if time: 30/37 easy....

$\hookrightarrow$ 22/37

(1.4) §§ BLOCK MULTIPLICATION:

THM 1) $BA = B[C_1 \; C_2 \; \cdots \; C_n] = [BC_1 \; BC_2 \; \cdots \; BC_n]$,

where B is $k \times m$; A is $m \times n$ given by $A = [C_1 \; C_2 \; \cdots \; C_n]$.

2) $$AX = [C_1 \; C_2 \; \cdots \; C_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{x_1 C_1 + x_2 C_2 + \cdots + x_n C_n}_{\text{linear combo.}},$$

where $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is $\boxed{n \times 1}$.

MORE IS TRUE: SAY: $A = \left[\begin{array}{c|c} P & X \\ \hline 0 & Q \end{array}\right]$; $B = \left[\begin{array}{c} U \\ \hline V \end{array}\right]$, then

$AB = \begin{bmatrix} PU + XV \\ QV \end{bmatrix}$ if the blocks are compatible for multiplication.

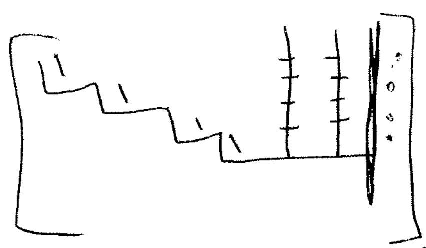
DO: 23/37

DO 25a/37

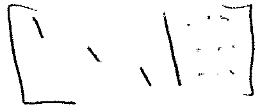
$$\begin{pmatrix} I & X \\ -Y & I \end{pmatrix} \begin{pmatrix} I & 0 \\ Y & I \end{pmatrix} = \begin{pmatrix} I + XY & X \\ 0 & I \end{pmatrix}$$

31/37 easy

16/37: 4×6, rank 4 $\Rightarrow$ YES: In RREF there is a leading 1 in every Row:



17/37 $AX = 0 \Rightarrow BAX = B0 \Rightarrow IX = 0 \Rightarrow X = 0$, contradiction

15/37 RREF of our setting is:  $\Rightarrow$ Uniqueness.

TRY yourself: 30, 22/37.

ONE Application from 1.2 : NETWORK FLOW 12/21

$$\begin{cases} \overset{\text{IN}}{f_4 + 70} = \overset{\text{OUT}}{f_1} \\ f_1 = 60 + f_2 \\ 50 + f_2 = f_3 \\ f_3 = 60 + f_4 \end{cases} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 70 \\ 1 & -1 & 0 & 0 & 60 \\ 0 & 1 & -1 & 0 & -50 \\ 0 & 0 & 1 & -1 & 60 \end{array}\right] \xrightarrow{R_2 - R_1}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 70 \\ 0 & -1 & 0 & 1 & -10 \\ 0 & 1 & -1 & 0 & -50 \\ 0 & 0 & 1 & -1 & 60 \end{array}\right] \xrightarrow{R_3 + R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 70 \\ 0 & -1 & 0 & 1 & -10 \\ 0 & 0 & -1 & 1 & -60 \\ 0 & 0 & 1 & -1 & 60 \end{array}\right]$$

$$\xrightarrow[(-1)R_2]{R_4 + R_3} \begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 70 \\ 0 & 1 & 0 & -1 & 10 \\ 0 & 0 & -1 & 1 & -60 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \end{array} \Rightarrow$$

$f_4 = t$, t scalar

$-f_3 = -60 - f_4$

$f_3 = 60 + t$; $f_2 = 10 + f_4 = 10 + t$; $f_1 = 70 + f_4 = 70 + t$.

Since $t \geq 0 \Rightarrow \boxed{f_1}$

1.5 MATRIX INVERSES

Def: Let A be a square mx. A mx. C is called an INVERSE of A if $AC = I$ and $CA = I$

Def A square mx. that has an inverse is called invertible.

EXC: DO 1/47.

FACT: The inverse is UNIQUE (when it exists).

NOTATION The inverse of A is denoted by A^{-1}! $A^{-1}A = AA^{-1} = I$

FORMULA: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

---

§ INVERSES and SYSTEMS

THM Consider a ~~the~~ system in n variables and of n equations:

$$AX = B.$$

If the $n \times n$ coefficient mx. A is invertible, the system has the unique solution: $X = A^{-1}B$

WHY? $A^{-1}(AX) = A^{-1}B \Rightarrow IX = A^{-1}B \Rightarrow \boxed{X = A^{-1}B}$

DO: 7 a/47 easy....

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§ The MATRIX InVerSion Algorithm

See pg 42

- If A is a square mx. and there is a sequence of Row op. that transforms $A \to I$, then A is invertible. Moreover, the same sequence transforms $I \to A^{-1}$.
- PLAN: $[A \mid I] \xrightarrow{\text{Row op}} [I \mid A^{-1}]$ (if A is inv.)

DO 2 e/47

---

§ PROPERTIES

1. I is invertible, and $I^{-1} = I$
2. if A is invertible, then A^{-1} is invertible and $(A^{-1})^{-1} = A$
3. if A, B are invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

(4) If $A_1, A_2, \ldots, A_k$ are all invertible, then $A_1 \cdot A_2 \cdots A$ is invertible, and $(A_1 \cdot A_2 \cdots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1} A_1^{-1}$.

(5) if A is invertible, then A^k is invertible for $k \geq 1$, and

$$(A^k)^{-1} = (A^{-1})^k$$

(6) If A is invertible, then A^T is <u>inv.</u> and $(A^T)^{-1} = (A^{-1})^T$.

(7) if A is invertible, $c \neq$ NOT ZERO, then cA is invertible, and $(cA)^{-1} = c^{-1} A^{-1}$.

Give a proof of one of them: request....

COR: A square, then: A is inv $\Leftrightarrow$ A^T is inv.

DO: 5c | 47 !!

PARTICULAR CASES: (Def) A square mx is called <u>UPPER triangular</u> if every entry below the main diagonal is zero.

(Def) A mx. A is <u>LOWER TRIANGULAR</u> if every entry above the main diagonal is zero.

(THM) Let A be a triangular (U or L) mx.

(1) A is invertible $\Leftrightarrow$ no entry on the main diagonal is 0.

(2) if A is Upper (Lower) triangular then A^{-1} is Upper (Lower) triangular

$\rightarrow$ GIVE IDEA of proof!!

EXP: $\begin{pmatrix} 3 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & \bullet \end{pmatrix}$ IS NOT INV

§ CONDITION for invertibility

(THM) The following conditions are equivalent for a square mx:

(1) A is invertible

(2) $AX=0$ has ONLY the trivial solution

(3) $AX=B$ has a solution X for each choice of column B.

(4) A can be transformed into I by Row oper.

(5) There is a mx. C such that $AC=I$.

DO: 19/48 ; 14/48 ; 16/48

13/48