CROSS PRODUCT

- Def: $\vec{v} \times \vec{w} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \times \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1 z_2 - y_2 z_1 \\ -(x_1 z_2 - x_2 z_1) \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$

PROPERTIES:

(1°) $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$ AND $\vec{w} \cdot (\vec{v} \times \vec{w}) = 0$

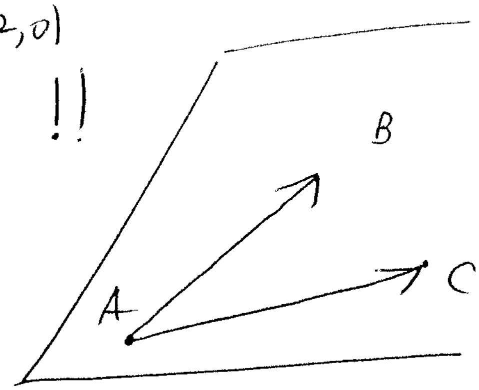
so: $\vec{v}, \vec{w} \perp \vec{v} \times \vec{w}$

(2°) $\vec{v} \times \vec{w} = \vec{0} \iff \vec{v}, \vec{w}$ are parallel

---

DO 13/159 a

$A(3,1,2)$; $B(5,-1,3)$; $C(-4,2,0)$



SOL: $\vec{AB}, \vec{AC}$ Are in the plane !!

We need a normal:

- $\vec{n} = \vec{AB} \times \vec{AC} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} -7 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -(3) \\ -12 \end{pmatrix}$

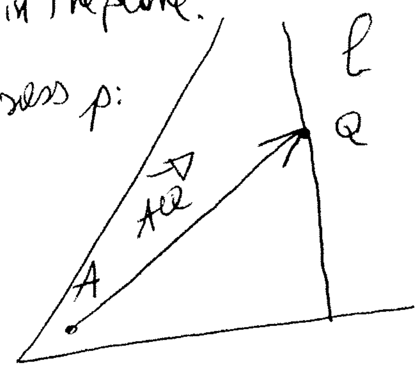
Hence the eq. of the plane is: $\begin{pmatrix} 3 \\ -3 \\ -12 \end{pmatrix} \cdot \begin{pmatrix} x-3 \\ y-1 \\ z-2 \end{pmatrix} = 0$ (since A is in the plane)

We get: $\boxed{x - y + 4z = 10}$

DO 13 g /159

Containing $A(2,1,-1)$ and the line: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$

SOL: $t = 0 \Rightarrow Q \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ is in the plane, hence $\vec{AQ}$ is in the plane.



$\vec{AQ} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. To get a normal, we use cross p:

- $\vec{n} = \vec{AQ} \times \vec{d} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix}$

The eq. is $\begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x-2 \\ y-1 \\ z+1 \end{pmatrix} = 0$, So: $\boxed{-3x - 2y - z = -7}$

3.5

CROSS PRODUCT

Th (Algebraic Properties of cross product)

- 1° $\vec{v}\times\vec{w}$ is a vector ; 2° $\vec{v}\times\vec{0}=\vec{0}=\vec{0}\times\vec{v}$; 3° $\vec{v}\times\vec{v}=\vec{0}$;

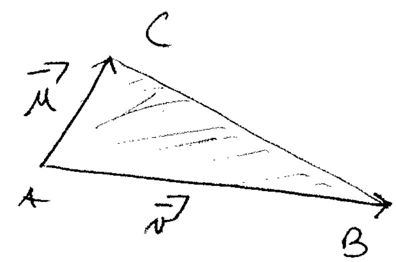
4° $\vec{w}\times\vec{v}=-(\vec{v}\times\vec{w})$ 5° $(a\vec{v})\times\vec{w}=a(\vec{v}\times\vec{w})=\vec{v}\times(a\vec{w})$; $a\in$

6° $\vec{v}\times(\vec{u}+\vec{w})=\vec{v}\times\vec{u}+\vec{v}\times\vec{w}$, 7° $(\vec{u}+\vec{w})\times\vec{v}=\vec{u}\times\vec{v}+\vec{w}\times\vec{v}$.

Th If θ is the angle between the vectors $\vec{v},\vec{w}$, then $\|\vec{v}\times\vec{w}\|=\|\vec{v}\|\cdot\|\vec{w}\|\cdot\sin\theta$ = Area of Parallelogram determined by $\vec{v},\vec{w}$:

DO ; 1a / 179 Area of $\triangle ABC$:

$A(1,-1,2)$; $B(3,3,2)$; $C(5,0,-4)$



SOL: $A=\frac{1}{2}\|\vec{u}\times\vec{v}\|$ where

So: $A=\frac{1}{2}\left\|\begin{pmatrix}4\\1\\-6\end{pmatrix}\times\begin{pmatrix}2\\4\\0\end{pmatrix}\right\|=$

$=\frac{1}{2}\left\|\begin{pmatrix}24\\-12\\14\end{pmatrix}\right\|=\frac{1}{2}\sqrt{(24)^2+(-12)^2+(14)^2}=\sqrt{229}.$

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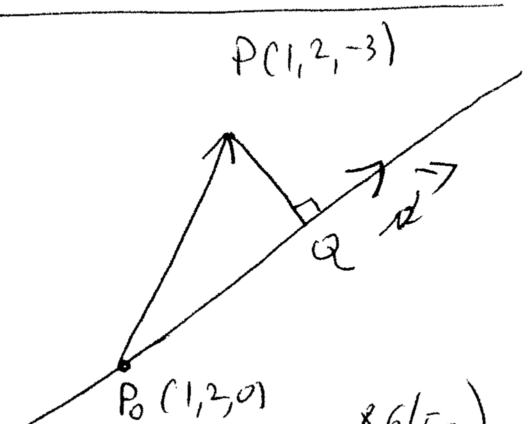
more on: DGD or Lecture 3

LAST one: 8a / 158 We note that:



$\overrightarrow{P_0Q}=\text{proj}_{(\vec{d})}(\overrightarrow{P_0P})=\frac{\overrightarrow{P_0P}\cdot\vec{d}}{\|\vec{d}\|^2}\cdot\vec{d}=$

$=\frac{\begin{pmatrix}0\\0\\-3\end{pmatrix}\cdot\begin{pmatrix}-3\\5\\4\end{pmatrix}}{50}\vec{d}=\frac{-12}{50}\begin{pmatrix}-3\\5\\4\end{pmatrix}.$

- if $Q(\alpha,\beta,\gamma)$ $\Rightarrow$ $\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=\begin{pmatrix}1\\2\\0\end{pmatrix}+\frac{-12}{50}\begin{pmatrix}-3\\5\\4\end{pmatrix}=\begin{pmatrix}86/50\\40/50\\-48/50\end{pmatrix}$

NOW: $\|PQ\|=\sqrt{\left(\frac{86}{50}-1\right)^2+\left(\frac{40}{50}-2\right)^2+\left(-\frac{48}{50}+3\right)^2}=\dots\dots$

Chapter 1 (LINEAR EQUATIONS and MATRICES)

§ 1.1 MATRICES

Def A rectangular ARRAY of #s is called a *matrix*. The numbers are called the entries of the mx.

EXP:

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Rows, Columns

Def A mx. with m Rows and n Columns is called an $m \times n$ matrix, (OR: The mx. has SIZE $m \times n$)

2×3 $\qquad$ 1×2 (Row mx) $\qquad$ 4×1 (Column mx)

NOTE: Each entry of a mx. is located by the Row and Colum in which it lies.
The entry in Row i and Column j is called the (i, j)-entry of the mx.

EXP: The $(2,3)$-entry of A is 2; The $(1,2)$-entry of B is -5
The $(3,1)$-entry of C is 3.

DISPLAY if A has size $m \times n$, if the (i,j)-entry of A is denoted by a_{ij}, then A is displayed:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{OR} \quad A = [a_{ij}]$$

Def: 2 matrices are equal if they have the same size AND if the corresponding entries are equal.

EXP. 1°) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \neq \begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $2 \neq -2$

2°) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix}$ Different sizes!!!

OPERATIONS: $\boxed{\text{MATRIX ADDITION}}$ Def if A, B are 2 matrices of the same size, their SUM $A+B$ is the mx. of the same size obtained by ADDING corresponding entries.

So: $A = [a_{ij}]$, $B = [b_{ij}] \Rightarrow A+B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$

Similarly: $\boxed{\text{The Difference}}$: $A - B = [a_{ij}] - [b_{ij}] = [a_{ij} - b_{ij}]$

EXP: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, $B = \begin{pmatrix} -1 & -3 & -2 \\ 1 & 2 & 3 \end{pmatrix} \Rightarrow A+B = \begin{pmatrix} 0 & -1 & -1 \\ 5 & 7 & 9 \end{pmatrix}$;

$A - B = \begin{pmatrix} 2 & 5 & 5 \\ 3 & 3 & 3 \end{pmatrix}$.

NOTATION
- if $A = [a_{ij}]$ is a mx, $-A = [-a_{ij}]$.
- The $m \times n$ mx in which every entry is 0 is called the ZERO mx, O.

Th 1°) $A+B = B+A$, 2°) $A + (B+C) = (A+B) + C$; 3°) $O + A = A$
4°) $A + (-A) = O$, where A, B, C, O are of the same size.

$\boxed{\text{SCALAR MULTIPLICATION}}$

Def If $A = [a_{ij}]$ has size $m \times n$; if c is a #, then the $\boxed{\text{scalar product}}$ cA is obtained by multiplying each entry of A by c:

$cA = c[a_{ij}] = [c \cdot a_{ij}]$.

NOTE: SAME SIZE!!

EXP: $(-3)\cdot\begin{bmatrix}1 & 2 & 3\\ 4 & 0 & 5\end{bmatrix} = \begin{bmatrix}-3 & -6 & -9\\ -12 & 0 & -15\end{bmatrix}$.

Th 1°) $c(A+B) = cA + cB$, 2°) $(c+d)A = cA + dA$; 3°) $c(dA) = (cd)A$; 4°) $1\cdot A = A$. (here c, d are #s)

Th $\boxed{\text{if } cA = O, \text{ then either } c = 0 \text{ or } A = O.}$

(DO the proof!!!)

TRANSPOSITION:

Def If $A = [a_{ij}]$ is an $m\times n$ mx., the main diagonal of A consists of ENTRIES $a_{11}, a_{22}, a_{33}, \ldots$

EXP:

$$\begin{pmatrix} a_{11} & & & \\ & a_{22} & & \\ & & a_{33} & \\ & & & \ddots \end{pmatrix}$$

Def $A_{m\times n}$ is called a square mx if # Rows = # of Columns.

Def: Let A be an $m\times n$ mx. Then the transpose of A, A^T, is the $\boxed{n\times m}$ mx. whose columns are formed from the corresponding rows of A:

$A = [a_{ij}] \longrightarrow A^T = [b_{ij}]$ where $b_{ij} = a_{ji}$

T (Properties) 1°) $(A^T)^T = A$; 2°) $(cA)^T = c(A^T)$; 3°) $(A+B)^T = A^T + B^T$

Def: A mx A is called symmetric if $A = A^T$ (it is a square mx).

DO 8 a/g T or F 6/9

Lecture 5 - TUE - JAN : $\boxed{1.2}$ LINEAR EQUATIONS

Q: what is the eq. of a line?

A: $ax+by=c$.

Generalization: Def A linear equation in variables $x_1, x_2, \ldots, x_n$ is an eq. of the form: $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$, where:

- $a_1, a_2, \ldots, a_n$ are #s, called the coefficients
- b is a #, called the constant term (free term).

EXP: $-x_1 + 3x_2 + \frac{1}{2}x_7 = -x_5$ ||| EXP $x_1x_2 + x_2 - x_4 = 0$

DEF $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is called the mx. of variables.

DEF: A set of #s $s_1, s_2, \ldots, s_n$ is called a solution of $a_1x_1 + \ldots + a_nx_4 = b$ if $a_1s_1 + a_2s_2 + \ldots + a_ns_4 = b$.

We SAY: $X = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$ is a solution of the eq. above.

EXP: $X = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$ is a solution of the eq. $x_1 - 2x_2 + 3x_3 + x_4 = -3$

The same for $Y = \begin{pmatrix} 1 \\ 4 \\ \vdots \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is NOT a solution!!

---

§ SYSTEMS:

Def: A finite collection of linear equations is called a system of linear equations.

Def A solution to $\boxed{\text{every}}$ equation in the system is called A SOLUTION of the system.

GOAL: FIND ALL solutions (of a given linear system)

DEF: A system is
- CONSISTENT: if it has one or more solutions
- INCONSISTENT: if it has NO SOLUTIONS

EXP: $\begin{cases} x + y = 1 \\ x - z = 2 \\ y + z = 1 \end{cases}$

SOL: Add the last 2 eqs: $x+y=3$. But $x+y=1$. So $3=1$, (F)
NO SOL; INCONSISTENT.

DO EXC. 1 a/20

(DEF) Given a system, the coefficients of the variables form a mx, called the coefficient matrix.

EXP $\begin{cases} x_1 + 2x_2 - 3x_3 - \frac{1}{2}x_4 = -3 \\ 2x_1 + x_2 + x_4 = 1 \end{cases} \rightsquigarrow \begin{bmatrix} 1 & 2 & -3 & -1/2 \\ 2 & 1 & 0 & 1 \end{bmatrix}$

(DEF) Given a system, the augmented mx. is just the coefficient mx. augmented by the column of constant term

$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & -1/2 & -3 \\ 2 & 1 & 0 & 1 & 1 \end{array}\right]$$

$\Rightarrow$ By manipulating this mx) we can get ALL solution of the system (if any).

(Def) 2 linear systems are $\boxed{\text{equivalent}}$ if they have the same set of solutions

(IDEA:) Replace the given system by an equivalent one, easier to solve.

EXP: $\begin{cases} x_1 - 2x_2 + 3x_3 + x_4 = -3 \\ 2x_1 - x_2 + 3x_3 - x_4 = 0 \end{cases}$

SOL:
- ELIMINATE x_1 from Eq. 2 (DO: Eq2 - 2* Eq.1) $\begin{cases} x_1 - 2x_2 + 3x_3 + x_4 = -3 \\ 3x_2 - 3x_3 - 3x_4 = 6 \end{cases}$
- Multiply by 1/3 Eq.2 $\begin{cases} x_1 - 2x_2 + 3x_3 + x_4 = -3 \\ x_2 - x_3 - x_4 = 2 \end{cases}$
- ELIMINATE x_2 from Eq. 1 (Do: Eq1 + 2xEq2) $\begin{cases} x_1 \quad + x_3 - x_4 = 1 \\ x_2 - x_3 - x_4 = 2 \end{cases}$

NOW: IT IS EASY to SOLVE: $X_3 = s$, $X_4 = t$, s, t scalars.

$X_2 = 2 + t + s$; $X_1 = 1 - s + t$. They 'WORK' for the ORIGINAL SYSTEM.

→ NOTE By performing the Above operations on Eqs., the set of solutions DID NOT change!!!

DEF: ELEMENTARY OPERATIONS

(I) Interchange 2 equations; (II) Multiply an equation by a nonzero #; (III) Add a multiple of one Eq. to a different Eq.

THM if an elementary operation is performed on a system, the resulting system is equivalent to the original system

FROM now ON: perform operations on a mx!!! to get sol of sys!

DEF: (I) Interchange 2 Rows;
(II) Multiply a Row by a nonzero #.
(III) Add a multiple of one Row to a different Row.

They Are Called: ELEMENTARY ROW OPERATIONS.

EXP: 4d/20 (FARA STATEMENT)

$$\left[\begin{array}{ccc|c} 2 & 2 & -3 & 1 \\ 1 & 0 & 1 & 5 \\ 3 & 4 & -7 & -3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 2 & 2 & -3 & 1 \\ 3 & 4 & -7 & -3 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 2 & -5 & -9 \\ 0 & 4 & -10 & -18 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 2 & -5 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & -5/2 & -9/2 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \end{array}$$

EASY to SOLVE:

$X_3 = s$, s-scalar; $X_2 = \frac{5}{2} s - \frac{9}{2}$; $X_1 = 5 - s$.

Done till here

GAUSSIAN Elimination

Def (REF) A mx. is in <u>Row Echelon Form</u> if:

(1) ALL zero Rows are at the bottom.

(2) The first non-zero ~~Row~~ entry from the left in each non-zero Row is a 1, called a leading 1 for that Row.

(3) <u>Each leading</u> 1 is to the <u>Right</u> of all leading 1's in the Rows above it

DEF (<u>RREF</u>) A row echelon mx is in Reduced Row Echelon Form if:

(4) Each leading 1 is the only non-zero entry in its column.

<u>EXP</u>: $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$; $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$; $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

The GAUSSIAN algorithm is the(a) sequence of Row operations that transforms a given mx. into RREF.

PG 16

DEF: The variables corresponding to the leading 1's are called: <u>LEADING VARIABLES</u>. (EXP...)

<u>PLAN (to solve systems)</u> (1) Construct the augmented mx, Row Red

(2) <u>Assign the non-leading variables AS parameters</u> (s, t, ...)

(3) Use the (R) REF to solve for leading variables in terms of parameters.

DO 4 f /20 BACK SUBSTITUTION

§ RANK of a mx

FACT The RREF of a mx. A is uniquely determined by A.

FACT The # of leading 1's is the same in any REForm.

This # is called : the rank of A.

THM3 If a system with m eq; n variables, has an augmented mx with rank r, there are n-r parameters.

THM4 A system can have: (1) NO SOL
(2) A unique sol
(3) ∞-ly many sol

(1) : [0 0 0 . . . 0 | b] b # NOT zero

(2) : ONLY leading var.

(3) : At least one non-leading var.

// ; X ; / DO 9 a, c / 21

If time 7/21

Lecture 6 - THU - 22^{nd} JAN - 2009

DEF: (REF) A mx. is in Row Echelon Form if:

① ALL zero Rows are at the bottom.

② The first non-zero entry from the left in each nonzero Row is a 1, called a leading 1 for that Row.

③ Each leading 1 is to the right of all leading 1's in the Rows above it.

DEF: (RREF) A row echelon form mx. is in Reduced Row Echelon Form if:

④ Each leading 1 is the only non-zero entry in its column.

EXS: $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$; $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$; $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

The GAUSSIAN algorithm is a sequence of Row Operations that transforms a given mx. into RREF: see page: 16.

BACK to SYSTEMS: ① The variables corresponding to the leading 1's are called: LEADING VARIABLES; (EXP);

PLAN (to solve systems) ① Construct the augmented mx and Row Reduce; ② Assign the non-leading variables AS parameters; ③ USE the (R)REF to solve for leading variables in terms of parameters.

| DO: 4 f/20 / BACK SUBSTITUTION

§ RANK of A MATRIX

FACT The RREF of a mx. A is uniquely determined by A.

FACT The # of leading 1's is the same in any REForm.

This # is called the rank of A.

THM3 if a SYSTEM with m eq., n variables, has an augmented mx. with rank r, then there are n-r parameters.

THM4 A system can have either (1) NO SOL (2) unique SOL or (3) infinitely many SOL

(1) $\Leftrightarrow$ $[0\ 0 \ldots 0 \mid b]$; $b \neq 0$

(2) $\Leftrightarrow$ ONLY leading variables

(3) $\Leftrightarrow$ At least one non-leading var

Do 9 a, c /21 (+) 7/21

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§ 1.3 HOMOGENEOUS SYSTEMS

Def A linear system is called homogeneous if all the constant / free terms are zero.

EXP:
$$\begin{cases} 3x - y + z + w = 0 \\ 2x + 2y + 3w = 0 \\ x + w = 0 \end{cases}$$

NOTE: Any homogeneous system has the trivial sol. (so: consistent...)

$x_1 = 0,\ x_2 = 0 \quad x_n = 0$

Q: Given a homogeneous system, DOES it have a non-trivial solution?

THM/22 if a homogeneous system has more variables than equations, then it has nontrivial solutions.

IDEA: $\left(\begin{array}{cccccccc|c} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \underbrace{\cdot \quad \cdot \quad \cdot \quad \cdot}_{\text{non-leading}} & & & & 0 \end{array}\right)$

non-leading → so-ly many sol

§ BASIC SOLUTIONS

Q: what are them?

EXP: 1 e/25

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -1 & 3 & 0 & 0 \\ -1 & 8 & 3 & 0 \end{array}\right] \xrightarrow[R_3+R_1]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 10 & 6 & 0 \end{array}\right] \to$$

$$\xrightarrow{R_3-2R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 3/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

L L N.Leading
x_1 x_2 x_3

$x_3 = t$; t scalar

$x_2 = -3/5\, t$

$x_1 = -2x_2 - 3x_3 = -2(-3/5\, t) - 3t = \left(\frac{6}{5} - 3\right)t = -\frac{9}{5}t$

So: $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -9/5\, t \\ -3/5\, t \\ t \end{pmatrix} = t \begin{pmatrix} -9/5 \\ -3/5 \\ 1 \end{pmatrix}$

$X_1 = \begin{pmatrix} -9/5 \\ -3/5 \\ 1 \end{pmatrix}$ (for $t=1$) is a specific solution, called BASIC solution.

Property: Any solution is a multiple of it!!

One more exp: 1 h/25

1.4./25

$$\left(\begin{array}{ccccc|c} 1 & 1 & -2 & 3 & 2 & 0 \\ 2 & -1 & 3 & 4 & 1 & 0 \\ -1 & -2 & 3 & 0 & 1 & 0 \\ 3 & 0 & 1 & 7 & 3 & 0 \end{array}\right) \xrightarrow[\substack{R_3+R_1 \\ R_4-3R_1}]{R_2-2R_1} \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 3 & 2 & 0 \\ 0 & -3 & 7 & -2 & -3 & 0 \\ 0 & -1 & 1 & 3 & 3 & 0 \\ 0 & -3 & 7 & -2 & -3 & 0 \end{array}\right]$$

$$\xrightarrow{R_4-R_2} \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 3 & 2 & 0 \\ 0 & -3 & 7 & -2 & -3 & 0 \\ 0 & -1 & 1 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2-3R_3} \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 3 & 2 & 0 \\ 0 & 0 & 4 & -11 & -12 & 0 \\ 0 & -1 & 1 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2\leftrightarrow R_3} \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 3 & 2 & 0 \\ 0 & 1 & -1 & -3 & -3 & 0 \\ 0 & 0 & 4 & -11 & -12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{4}R_3} \begin{array}{c} \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 3 & 2 & 0 \\ 0 & 1 & -1 & -3 & -3 & 0 \\ 0 & 0 & 1 & -\frac{11}{4} & -\frac{12}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \end{array}$$

$x_4 = t$; $x_5 = s$

$x_3 = \frac{11}{4}t + \frac{12}{4}s$; $x_2 = \frac{11}{4}t + \frac{12}{4}s + 3t + 3s = t\left(\frac{23}{4}\right) + s\left(\frac{24}{4}\right)$

$x_1 = -t\left(\frac{23}{4}\right) - s\left(\frac{24}{4}\right) + \frac{22}{4}t + \frac{24}{4}s - 3t - 2s$

$= t\left(-\frac{13}{4}\right) + s(-2)$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -13/4\,t - 2s \\ 23/4\,t + 6s \\ 11/4\,t + 12/4\,s \\ t \\ s \end{pmatrix} = t\begin{pmatrix} -13/4 \\ 23/4 \\ 11/4 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -2 \\ 6 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

$X_1 = \begin{pmatrix} -13/4 \\ 23/4 \\ 11/4 \\ 1 \\ 0 \end{pmatrix}$; $X_2 = \begin{pmatrix} -2 \\ 6 \\ 3 \\ 0 \\ 1 \end{pmatrix}$ are Basic sol; why?

Any solution has the form $\boxed{tX_1 + sX_2}$

The same is true for any homogeneous system: to express every solution !! in a certain way.

Def: if $X_1, X_2, \dots, X_k$ are columns, the expression

$s_1 X_1 + s_2 X_2 + \dots + s_k X_k$, where $s_1, \dots, s_k$ are #s, is called a linear combination of $x_1, x_2, \dots, x_k$.

So: Every solution of a homogeneous system can be expressed as a linear combination of the BASIC SOLUTIONS (produced by Row Reduction)

(THM) If a homogeneous system in n variables has the augmented mx with rank r, then

(1) There are $n-r$ basic solutions

(2) Every solution is a linear combination of the basic solutions.