4.5 ORTHOGONALITY

Def: 1) if $\left\{ \begin{array}{l} X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ is in } \mathbb{R}^n \\ Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \text{ is in } \mathbb{R}^n \end{array} \right.$, we define their <u>DOT PRODUC</u>

as follows: $X \cdot Y = \underbrace{x_1 y_1 + x_2 y_2 + \dots + x_n y_n}_{\text{a number}} = X^T Y$

2) The length of X is $\|X\| = \sqrt{X \cdot X} =$

$= \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$.

<u>Exp:</u> if $X = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 3 \end{pmatrix}$ and $Y = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 2 \\ -3 \end{pmatrix} \Rightarrow X \cdot Y = -2 - 1 + 0 + 2 - 9 = -10$

and $\|X\| = \sqrt{1+1+1+9} = \sqrt{12}$

<u>PROPERTIES:</u> ① $X \cdot Y = Y \cdot X$; ② $X \cdot (Y + Z) = X \cdot Y + X \cdot Z$

③ $(aX) \cdot Y = a(X \cdot Y) = X \cdot (aY)$, a #

④ $\|X\|^2 = X \cdot X$; ⑤ $\|X\| \geq 0$; $\|X\| = 0 \iff X = 0$.

⑥ $\|aX\| = |a| \, \|X\|$, a #. <u>(same as in $\mathbb{R}^2$ or $\mathbb{R}^3$)</u>

Def: ① 2 vectors X, Y in $\mathbb{R}^n$ are called ORTHOGONAL if $X \cdot Y = 0$

② A set $\{X_1, X_2, \dots, X_n\}$ is called an ORTHOGONAL SET if :

$$\begin{cases} X_i \cdot X_j = 0 & \text{for all } i \neq j \\ X_i \neq 0 & \text{for all } i. \end{cases}$$

3) A set $\{X_1, X_2, \dots, X_n\}$ is called orthoNORMAL if it is orthogonal and $\|X_i\| = 1$ for all $i = 1, 2, \dots, n$.

EXP: $\{E_1, E_2, \dots, E_n\}$ where $I_n = [E_1 \ E_2 \dots E_n]$

SOL: Indeed $\|E_i\| = \sqrt{0 + 0 + \dots + 1^2 + 0 + \dots + 0} = \sqrt{1} = 1$,

since $E_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th}$ pos.

ALSO: $E_i \cdot E_j = 0 + 0 + \dots + 0 = 0$ for all $i \neq j$.

(THM) Every ORTHOGONAL set of vectors is linearly Independent.

$\boxed{\text{Pf:}}$ Suppose $t_1 X_1 + \dots + t_k X_k = 0$, (where $\{X_1, X_2, \dots, X_k\}$ is a ORTHOGONAL SET); then:

$$0 \overset{!}{=} X_1 \cdot 0 = X_1 \cdot (t_1 X_1 + \dots + t_k X_k) =$$
$$= t_1 (X_1 \cdot X_1) + t_2 \underbrace{(X_1 \cdot X_2)}_{0} + \dots + t_k \underbrace{(X_1 \cdot X_k)}_{0}$$
$$= t_1 \|X_1\|^2 \Rightarrow t_1 = 0;$$ The same with the other t_i's.

We are interested in ORTHOGONAL BASIS!

IDEA: from a basis $\xrightarrow{\text{ALGORITHM}}$ ORTHOGONAL BASIS:

(THM) GRAM-SCHMIDT algorithm

Let $\{X_1, X_2, \dots, X_m\}$ be a basis of a subspace U of $\mathbb{R}^n$. Construct:

$F_1 = X_1$;

$$F_2 = X_2 - \frac{X_2 \cdot F_1}{\|F_1\|^2} \cdot F_1$$

$$F_3 = X_3 - \frac{X_3 \cdot F_1}{\|F_1\|^2} F_1 - \frac{X_3 \cdot F_2}{\|F_2\|^2} F_2$$

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$$F_k = X_k - \frac{X_k \cdot F_1}{\|F_1\|^2} F_1 - \frac{X_k \cdot F_2}{\|F_2\|^2} F_2 - \dots - \frac{X_k \cdot F_{k-1}}{\|F_{k-1}\|^2} F_{k-1}$$

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THEN: $\{F_1, F_2, \dots, F_m\}$ is an ORTHOGONAL basis of U.

EXC: 7 c/ 221 $\left\{ X_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}; X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}; X_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$

SOL: $F_1 = X_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$$F_2 = X_2 - \frac{X_2 \cdot F_1}{\|F_1\|^2} \cdot F_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{1+1+1} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - 2/3 \\ 2/3 \\ 1 - 2/3 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \\ 1/3 \end{pmatrix}.$$

$$F_3 = X_3 - \frac{X_3 \cdot F_1}{\|F_1\|^2} F_1 - \frac{X_3 \cdot F_2}{\|F_2\|^2} \cdot F_2 =$$

$$= \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \frac{5/3}{\frac{1}{9} + \frac{4}{9} + \frac{1}{9}} \begin{pmatrix} 1/3 \\ 2/3 \\ 1/3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2/3 \\ -2/3 \\ 2/3 \end{pmatrix} - \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1/3 \end{pmatrix} \cdot \frac{5}{2} =$$

$$= \begin{pmatrix} 1 - 2/3 - \frac{5}{6} \\ 1 + 2/3 - 5/3 \\ 2 - \frac{2}{3} - \frac{5}{6} \end{pmatrix} = \begin{pmatrix} -1/2 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

FOR what is that GOOD?

4.6 PROJECTIONS

Def if U is a subspace of $\mathbb{R}^n$, the ORTHOGONAL COMPLEMENT $U^\perp$ is given by: $U^\perp = \{ X \text{ in } \mathbb{R}^n \mid X \cdot Y = 0 \text{ for all } Y \text{ in } U \}$.

THM: 1) $U^\perp$ is a SUBSPACE of $\mathbb{R}^n$!

2) if $U = \text{span}\{X_1, X_2, \ldots, X_k\}$, then $U^\perp = \{ X \mid X \cdot X_i = 0 \text{ for all } i = 1, 2, \ldots, k \}$.

Def: if $\{F_1, F_2, \ldots, F_m\}$ is an ORTHOGONAL basis of U, the vector $\operatorname{proj}_U(X) = \frac{X \cdot F_1}{\|F_1\|^2} \cdot F_1 + \frac{X \cdot F_2}{\|F_2\|^2} \cdot F_2 + \cdots + \frac{X \cdot F_m}{\|F_m\|^2} F_m$ is called projection of X on U (we assume $U \neq \{0\}$).

Do: 2 a/233   Let $F_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}$; $F_2 = \begin{pmatrix} 2 \\ 2 \\ -1 \\ 0 \end{pmatrix}$

$F_1 \cdot F_2 = 2 - 2 + 0 + 0 = 0 \Rightarrow \{F_1, F_2\}$ is ORTHOG. BASIS

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$$\operatorname{proj}_U(X) = \frac{X \cdot F_1}{\|F_1\|^2} F_1 + \frac{X \cdot F_2}{\|F_2\|^2} F_2 =$$

$$= \frac{\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}}{1+1+0+4} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix} + \frac{\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ -1 \\ 0 \end{pmatrix}}{4+4+1+0} \begin{pmatrix} 2 \\ 2 \\ -1 \\ 0 \end{pmatrix}$$

$$= \frac{0}{6} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix} + \frac{4}{9} \begin{pmatrix} 2 \\ 2 \\ -1 \\ 0 \end{pmatrix}$$

$$= \frac{4}{9} \begin{pmatrix} 2 \\ 2 \\ -1 \\ 0 \end{pmatrix}$$

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more IMP: EXP 4/226

(find $\operatorname{proj}_U(X) = ?$)

In this EXP/226, the Given basis
$\{X_1, X_2\}$ is NOT ORTHOG.
You have to apply the Gram-Schmidt Algorithm to get an ORTHOG. BASIS (say $\{F_1, F_2\}$), and then
Use the formula

$$proj_U(X) = \frac{X \cdot F_1}{\|F_1\|^2} \cdot F_1 + \frac{X \cdot F_2}{\|F_2\|^2} \cdot F_2$$

@ blackboard . . . .