Lecture: 2nd: APRIL: 2009;

EXC: Let A be an $n\times n$ mx. Show that: A is invertibl $\Longleftrightarrow$ 0 is not an eigenvalue of A.

SOL: Recall that λ is an eigenvalue (of A) $\Longleftrightarrow$ $\det(\lambda I - A) = 0$. So: 0 is an eigenvalue of A $\Longleftrightarrow$ $\det(0\cdot I - A) = 0$. So: 0 is an eigenvalue of A $\Longleftrightarrow$ $\det(-A) = 0$. So: 0 is an eigenvalue of A $\Longleftrightarrow$ $(-1)^n \det A = 0$. So: 0 is an eigenvalue of A $\Longleftrightarrow$ $\det A = 0$ $\Longleftrightarrow$ A is NOT invertible. Done

NO: 2.5!!

§2.8 SYSTEMS OF DIFFERENTIAL EQUATIONS

Q: what is a differential equation?
Q: what is a FIRST ORDER DIFFERENTIAL equations?

Lemma The solution of the differential equation:

$$f' = a\cdot f \quad , \text{where } a \text{ is a } \# \text{, is}$$

$$f(x) = c\cdot e^{ax}, \text{ for some } \# \ c.$$

SOL: Look at $f(x)\cdot e^{-ax} = g(x)$; $g'(x) = f'(x)\, e^{-ax} + f(x)(-a)e^{-ax} = e^{-ax}(f'(x) - a f(x)) = 0 \Rightarrow g'(x) = 0 \Rightarrow g(x) = c$ for some $\#$ c. So $f(x) = c\cdot e^{ax}$

What if we have more functions?

THE SYSTEM:

$$\begin{cases} f_1' = a_{11}f_1 + a_{12}f_2 + \cdots + a_{1n}f_n \\ f_2' = a_{21}f_1 + a_{22}f_2 + \cdots + a_{2n}f_n \\ \text{— — — — — — —} \\ f_n' = a_{n1}f_1 + a_{n2}f_2 + \cdots + a_{nn}f_n \end{cases}$$

NOTATION: $f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$; $f' = \begin{bmatrix} f_1' \\ f_2' \\ \vdots \\ f_n' \end{bmatrix}$; $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

IT can be written AS: $\boxed{f' = A\cdot f}$ ($\Leftarrow$ see lemma!)

Def: The system is called: a system of first order differential equations.

Thm (see lemma) Consider the system: $f' = Af$. If A is diagonalizable, then every solution of $f' = Af$ is given by $f(x) = c_1 X_1 e^{\lambda_1 x} + c_2 X_2 e^{\lambda_2 x} + \cdots + c_n X_n e^{\lambda_n x}$, where $c_1, c_2, \ldots, c_n$ are scalars; $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A; $X_1, X_2, \ldots, X_n$ are the corresponding eigenvectors.

DO: 1 a/132

$$\begin{cases} f_1' = 2f_1 + 4f_2 \\ f_2' = 3f_1 + 3f_2 \end{cases}; \quad \begin{cases} f_1(0) = 0 \\ f_2(0) = 1 \end{cases}$$

SOL: $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$; $f' = \begin{pmatrix} f_1' \\ f_2' \end{pmatrix}$; $f' = Af$; $A = \begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix}$

$c_A(x) = \det(xI_2 - A) = \det\left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix}\right) = \det\begin{pmatrix} x-2 & -4 \\ -3 & x-3 \end{pmatrix}$

$= (x-2)(x-3) - 12 = x^2 - 5x + 6 - 12 = x^2 - 5x - 6 =$

$= (x-6)(x+1) \Rightarrow \lambda_1 = 6; \quad \lambda_2 = -1$. (DIFFERENT (or DISTINCT) eigenvalues $\Rightarrow$ A is diagonalizable!!!!)

FOR $\lambda_1 = 6$, we solve $(6I_2 - A)X = 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $\left(\begin{array}{cc|c} 4 & -4 & 0 \\ -3 & 3 & 0 \end{array}\right) \Rightarrow$

$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x_2 = t; \quad x_1 = t, \quad t \# \Rightarrow X = \begin{pmatrix} t \\ t \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

choose $X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

FOR $\lambda_2 = -1$, we solve: $((-1)I_2 - A)X = 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $\left(\begin{array}{cc|c} -3 & -4 & 0 \\ -3 & -4 & 0 \end{array}\right) \Rightarrow$

$\Rightarrow \left(\begin{array}{cc|c} -3 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x_2 = t; \quad x_1 = \frac{4}{-3}t \Rightarrow X = \begin{pmatrix} -4/3\, t \\ t \end{pmatrix} = t\begin{pmatrix} -4/3 \\ 1 \end{pmatrix}$

t scalar.

choose $X_2 = \begin{pmatrix} -4/3 \\ 1 \end{pmatrix}$.

So $f(x) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{6x} + c_2\begin{pmatrix} -4/3 \\ 1 \end{pmatrix}e^{-x}$; To get c_1, c_2

Recall that $\begin{cases} f_1(0) = 0 \\ f_2(0) = 1 \end{cases} \Rightarrow \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow$

$c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^0 + c_2\begin{pmatrix} -4/3 \\ 1 \end{pmatrix}e^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow$

$c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -4/3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} 1 & -4/3 & 0 \\ 1 & 1 & 1 \end{array}\right) \xrightarrow{R_2 - R_1}$

$\left(\begin{array}{cc|c} 1 & -4/3 & 0 \\ 0 & \frac{7}{3} & 1 \end{array}\right) \Rightarrow c_2 = 3/7 \Rightarrow c_1 = 4/3 \cdot c_2 = \frac{4}{7}$

So $f(x) = \frac{4}{7}\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{6x} + \frac{3}{7}\begin{pmatrix} -4/3 \\ 1 \end{pmatrix}e^{-x}$

TRY yourself 1.C./132

2.4. Linear Dynamical Systems

Def A *dynamical system* is a sequence of columns V_0, V_1, V_2 ---- where the initial V_0 is known, and the other are determined by: $V_{k+1} = AV_k$ for $k \geq 0$; (A is a square mx).

The condition $V_{k+1} = AV_k$ is called a mx. recurrence.

Since V_0 is known $\Rightarrow$ $V_1 = AV_0$ can be found; $V_2 = AV_1 = AAV_0 = A^2V_0$ can be found; $V_3 = AV_2 = AA^2V_0 = A^3V_0$, ---

EX: Consider a population of BIRDS. Set $V_k = \begin{bmatrix} a_k \\ j_k \end{bmatrix}$, where a_k and j_k represents the # of adult and juvenile females present k years after the initial values a_0, j_0 were observed.

Assume they satisfy the equations: $\begin{cases} a_{k+1} = \alpha a_k + \beta j_k \\ j_{k+1} = m a_k \end{cases}$

where: α and β are ADULT and JUVENILE survival rates; m is the reproduction rate. We assume $\alpha, \beta, m \geq 0$.

So $A = \begin{bmatrix} \alpha & \beta \\ m & 0 \end{bmatrix}$: $V_{k+1} = \begin{bmatrix} a_{k+1} \\ j_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha & \beta \\ m & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_k \\ j_k \end{bmatrix}}_{V_k}$.

Def An eigenvalue λ of an $n \times n$ mx A is called a *dominant eigenvalue* of A if: $|\lambda| > |\mu|$ for all $\mu \neq \lambda$ eigenvalues.

(THM) Consider the dynamical system: $V_{k+1} = A V_k;\ k \geq 0$ where A, and V_0 are given; Assume that A is diagonalizable $n \times n$ mx with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and corresponding eigenvectors $X_1, X_2, \dots, X_n$ and let $P = [X_1\ X_2 \dots X_n]$. Then:

$$V_k = b_1 \lambda_1^k X_1 + b_2 \lambda_2^k X_2 + \dots + b_n \lambda_n^k X_n \ ; \ k \geq 0$$

where $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = P^{-1} V_0$.

MOREOVER if A has dominant eigenvalue λ_1, of multiplicity 1, then $V_k \approx \lambda_1^k b_1 X_1$ for sufficiently large k.

Interpretation:

$\boxed{\text{Say } \lambda_1 > 0}$

$\lambda_1 > 1 \Rightarrow$ pop. becomes very large as $k \nearrow$

$\lambda_1 = 1 \Rightarrow$ pop. stabilizes as $k \nearrow$

$\lambda_1 < 1 \Rightarrow$ pop becomes very small as $k \nearrow$ (extinction)

1/109 $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ 2 & 0 \end{pmatrix}$; $V_k = \begin{pmatrix} a_k \\ j_k \end{pmatrix}$ $k \geq 0$

SOL:

(1) $V_0 = \begin{pmatrix} 100 \\ 60 \end{pmatrix}$; $V_1 = A V_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 100 \\ 60 \end{pmatrix} = \begin{pmatrix} 70 \\ 200 \end{pmatrix}$

$V_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 70 \\ 200 \end{pmatrix} = \begin{pmatrix} 35 + 66.6 \\ 140 \end{pmatrix}$ .... etc...

(2) $c_A(x) = \det(xI_2 - A) = \det\begin{pmatrix} x - \frac{1}{2} & -\frac{1}{3} \\ -2 & x \end{pmatrix} =$

$= x^2 - \frac{1}{2}x - \frac{2}{3} \Rightarrow \lambda_1 = \dfrac{\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{8}{3}}}{2}$

$\lambda_2 = \frac{1}{2} - \sqrt{\frac{35}{4}}$

$$\Rightarrow \begin{cases} \lambda_1 = \frac{1}{4}\left[1 + \sqrt{\frac{35}{3}}\right] \approx 1.104 \\ \lambda_2 = \frac{1}{4}\left[1 - \sqrt{\frac{35}{3}}\right] \approx -0.604 \end{cases}$$

Since λ_1 is dominant and $> 1 \Rightarrow$ pop. becomes large as $K \nearrow$. (3) it is independent of initial pop.

(4) $V_K \approx b_1 \cdot \lambda_1^K \cdot X_1$

To GET X_1: solve $(\lambda_1 I_2 - A) X_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1.104 & 0 \\ 0 & 1.104 \end{pmatrix} - \begin{pmatrix} 0.500 & 0.333 \\ 2.000 & 0.000 \end{pmatrix} =$$

$$= \begin{pmatrix} 5.104 & -0.333 \\ -2.000 & 1.104 \end{pmatrix}$$

So $\left(\begin{array}{cc|c} 5.104 & -0.333 & 0 \\ -2.000 & 1.104 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 2.000 & -1.104 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$X_1 = \begin{pmatrix} a \\ b \end{pmatrix}$ $\quad b = t \ ; \ a = \frac{1.104}{2} t \quad$ choose $t = 2$

$\Rightarrow X_1 \approx \begin{pmatrix} 1.104 \\ 2 \end{pmatrix}$

Similarly $X_2 \approx \begin{pmatrix} -0.604 \\ 2 \end{pmatrix}$

So $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = P^{-1} V_0 = \begin{bmatrix} 1.104 & -0.604 \\ 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 100 \\ 60 \end{bmatrix} =$

$$\frac{1}{2.208+1.208}\begin{bmatrix}2 & 0.604\\ -2 & 1.104\end{bmatrix}\begin{bmatrix}100\\ 60\end{bmatrix}$$

$$=\frac{1}{3.416}\begin{bmatrix}200 + 36.24\\ -200 + 66.24\end{bmatrix} = \begin{pmatrix}69.16\\ -39.16\end{pmatrix}$$

So $V_K \approx (69.16)(1.104)^K \begin{pmatrix}1.104\\ 2\end{pmatrix} = \begin{cases} a_K \approx (1.104)^K\, 76.3\\ j_K \approx (1.104)^K\, 138.3\end{cases}$

So Done...

$\boxed{2/109}$ a) $A = \begin{pmatrix}\frac{3}{10} & \frac{1}{3}\\ 2 & 0\end{pmatrix}$; $V_0 = \begin{pmatrix}100\\ 50\end{pmatrix}$; $V_K = A^K V_0$

(b) $\lambda_1 = \frac{1}{60}\left(9+\sqrt{2481}\right) \approx 0.981$

$\lambda_2 = \frac{1}{60}\left(9-\sqrt{2481}\right) \approx -0.680$ because:

$$x I_2 - A = \begin{pmatrix}x-\frac{3}{10} & -\frac{1}{3}\\ -2 & x\end{pmatrix} \Rightarrow x^2 - \frac{3}{10}x - \frac{2}{3} = 0$$

$$\Rightarrow \lambda_1 = \frac{\frac{3}{10} + \sqrt{\frac{9}{100}+\frac{8}{3}}}{2} = \frac{\frac{3}{10} + \sqrt{\frac{27+800}{100\cdot 3}}}{2}$$

$$= \frac{3+\sqrt{\frac{827}{3}}}{20} = \frac{3+\frac{\sqrt{2481}}{3}}{20} = \frac{9+\sqrt{2481}}{60}$$

(c) $x_1 = \begin{pmatrix}\cdot\\ \cdot\end{pmatrix}$; $x_2 = \begin{pmatrix}\cdot\\ \cdot\end{pmatrix}$ as follows:

$$\left[\begin{array}{cc|c} 0.981 - 0.3 & -0.333 & 0 \\ -2 & 0.981 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0.648 & -0.333 & 0 \\ -2 & 0.981 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|c} 0.68 & -0.333 & 0 \\ 0 & 0 & 0 \end{array}\right]. \text{ If } x_1 = \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow$$

$$b = t \Rightarrow a = \frac{0.333}{0.68} t \Rightarrow_{t=2} X_1 \simeq \begin{pmatrix} 0.981 \\ 2 \end{pmatrix}$$

In the same way: $X_2 = \begin{pmatrix} -0.680 \\ 2 \end{pmatrix}$; $P = [X_1 \ \ X_2]$.

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c) $V_K = b_1 \lambda_1^K X_1 + b_2 \lambda_2^K X_2$, where

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \left(P^{-1} V_0\right) = \begin{bmatrix} 0.981 & -0.680 \\ 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 100 \\ 50 \end{bmatrix} =$$

$$\frac{1}{2(0.981) + 2(0.680)} \begin{bmatrix} 2 & 0.680 \\ -2 & 0.981 \end{bmatrix} \begin{bmatrix} 100 \\ 50 \end{bmatrix} =$$

$$= \begin{pmatrix} 70.439 \\ -45.439 \end{pmatrix}.$$ Just plug in now in

d) We need V_5, so by above...

$$V_5 = \begin{pmatrix} 58.288 \\ 141.21 \end{pmatrix}$$

(e) $\lambda_1 = 0.981$ is dominant

f) $V_K \approx (70.439)(0.981)^K \begin{pmatrix} 0.981 \\ 2 \end{pmatrix}$.

g) $V_{10} \approx \begin{pmatrix} 57.039 \\ 116.29 \end{pmatrix} \begin{matrix} \leftarrow a_{10} \\ \leftarrow j_{10} \end{matrix}$

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h) They DIFFER by $\begin{pmatrix} -0.65317 \\ 1.9211 \end{pmatrix}$

i) POP. becomes extinct because the dominant eigenvalue $\lambda_1 < 1$

j) NO, it will be extinct for every initial pop.

k) $\dfrac{j_K}{a_K + j_K} = 0.671 \Rightarrow 67.1\%$ are juveniles.