2.3 Lecture : TUE : 31^{ST} of MARCH

THM if an $n\times n$ mx. A has n distinct eigenvalues, then A is diagonalizable.

TERMINOLOGY: Let λ be an eigenvalue of A. The set null $(\lambda I - A)$ is called the eigenspace associated to λ.

§§ SIMILARITY

Def: A, B are called similar if $B = P^{-1}AP$ for some invertible mx. P. We write $A \sim B$.

NOTE: $A \sim A$; $A \sim B \Rightarrow B \sim A$; $A \sim B, B \sim C \Rightarrow A \sim C$.

THM if A, B are similar, then: (1) $\det A = \det B$; (2) $c_A(x) = c_B(x)$; (3) A, B have the same eigenvalues.

Pf: @ blackboard: easy!

NOTE: $A \sim B \Rightarrow \begin{cases} A^{-1} \sim B^{-1} & \text{(if } A, B \text{ are inv.)} \\ A^T \sim B^T & \\ A^k \sim B^k & , k \geq 0. \end{cases}$

---

12 a / 33 ; 6a NOT ; 11 a, b or c

2.5 COMPLEX NUMBERS

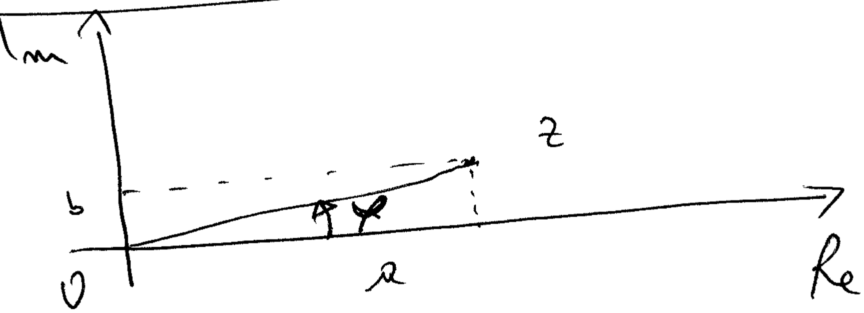
FACT: Let A be $n\times n$ (with real or complex #s as entries); Construct $c_A(x) = \det(xI_n - A)$; it is a polynomial of degree n. So by Fundamental Theorem of Algebra, it has $\boxed{n}$ complex roots (i.e. the roots are complex #s, they can be real/and/or complex, possibly some repeated!!)

So: Every $n \times n$ mx. A has n complex eigenvalues.

THM: if A is a symmetric ($A = A^T$) real mx, then any eigenvalue of A is real

---

POLAR FORM:



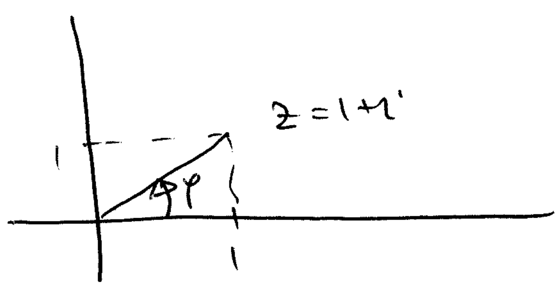
$z = a + bi$

$$\begin{cases} \cos\varphi = \dfrac{a}{|z|} \\ \sin\varphi = \dfrac{b}{|z|} \end{cases} \Rightarrow \quad a = |z|\cos\varphi \; ; \; b = (\sin\varphi)|z|.$$

So $z = |z|(\cos\varphi + i\sin\varphi)$, where $|z| \geq 0$; φ in radians. It is called the POLAR FORM of z.

DEF: $\boxed{e^{i\varphi} = \cos\varphi + i\sin\varphi}$ So: $\boxed{z = |z| \cdot e^{i\varphi}}$ (still POLAR form)

EXP:



Find the polar form of $z = 1 + i$

$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\cos\varphi = \frac{\pi}{4} \Rightarrow$

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\, e^{\frac{\pi}{4} i}$$

- MULTIPLICATION Rule: if $z = |z| e^{i\theta}$, $w = |w| e^{i\alpha}$ $\Rightarrow$

$$zw = |z||w| e^{i(\theta+\alpha)} \quad \text{or}$$

$$zw = \Big[|z| (\cos\theta + i\sin\theta)\Big]\Big[|w| (\cos\alpha + i\sin\alpha)\Big] =$$

$$= |zw| \Big(\cos(\theta+\alpha) + i\sin(\theta+\alpha)\Big).$$

- De Moivre's Theorem: if $z = |z| (\cos\alpha + i\sin\alpha)$

$$= |z| e^{i\alpha} \Rightarrow z^k = |z|^k e^{ik\alpha} \quad ; \quad k = 0, \pm 1, \pm 2, \ldots$$

Exp: Compute $(1+i)^6 = \left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^6$

$$= (\sqrt{2})^6 \left(\cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}\right) =$$

$$= 8\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = 8(0 - i) = \boxed{-8i}$$

THM ~~[illegible]~~ The Roots of the eq.

$$z^n = 1 \quad \text{are:} \quad e^{\frac{2\pi}{n}\cdot k\cdot i} \quad ; \quad k = 0, 1, 2, \ldots, n$$

Pf: Use De Moivre's Theorem . . . .

---

24 a/120 Solve $z^4 = -1$

SOL: So $z^4 = 1\cdot(\cos\pi + i\sin\pi)$

Suppose $z = |z|(\cos\alpha + i\sin\alpha) \Rightarrow \begin{cases} |z|^4 = 1 \\ 4\alpha = \pi + 2k\pi \; ; \; k \in \mathbb{Z} \end{cases}$

(2)

$\Rightarrow \alpha = \frac{\pi}{4} + \frac{k\pi}{2}$, $k \in \mathbb{Z}$; so k can be 0, 1, 2, 3; (all the other choices differ from one of these by a multiple of 2π, so no new roots).

So
$$\begin{cases} z_0 = (\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) \\ z_1 = \cos(\frac{\pi}{4}+\frac{\pi}{2}) + i\sin(\frac{\pi}{4}+\frac{\pi}{2}) \\ z_2 = \cos(\frac{\pi}{4}+\pi) + i\sin(\frac{\pi}{4}+\pi) \\ z_3 = \cos(\frac{\pi}{4}+\frac{3\pi}{2}) + i\cdot\sin(\frac{\pi}{4}+\frac{3\pi}{2}) \end{cases}$$

2.8 (SYSTEMS OF DIFFERENTIAL EQUATIONS)

THE EASY CASE:

Lemma: The solution of the differential eq.
$$f' = a\cdot f \quad \text{, where } a \text{ is a \#}$$
is $f(x) = c\cdot e^{ax}$, for some c #.

SOL: look at $f(x)\cdot e^{-ax} = g(x)$; $g'(x) = f'(x)\,e^{-ax} + f(x)(-a)\,e^{-ax} = e^{-ax}(f'(x) - af(x)) = 0$

$\Rightarrow g'(x) = 0 \Rightarrow g(x) = c$, c # $\Rightarrow f(x)e^{-ax} = c$

$\Rightarrow \boxed{f(x) = c\,e^{ax}}$

What if we have more functions?

THE SYSTEM:

$$\begin{cases} f_1' = a_{11} f_1 + a_{12} f_2 + \dots + a_{1n} f_n \\ f_2' = a_{21} f_1 + a_{22} f_2 + \dots + a_{2n} f_n \\ \dots \\ f_n' = a_{n1} f_1 + a_{n2} f_2 + \dots + a_{nn} f_n \end{cases}$$

if $f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$, let $f' = \begin{bmatrix} f_1' \\ f_2' \\ \vdots \\ f_n' \end{bmatrix}$; $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$

so: it can be re written AS:

$$f' = A \cdot f$$ ⇐ see the lemma!

D) The system above is called: a system of first order differential equations.

THM (see ALSO lemma) Consider the system: $f' = Af$. If A is diagonalizable, then every solution of $f' = Af$ is given by:

$$f(x) = c_1 X_1 e^{\lambda_1 x} + c_2 X_2 e^{\lambda_2 x} + \dots + c_n X_n e^{\lambda_n},$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A; $X_1, X_2, \dots, X_n$ are the corresponding eigenvectors; $c_1, c_2, \dots, c_n$ are scalars.

1ℓ/132 $\quad \begin{cases} f_1' = 2f_1 + 4f_2 \\ f_2' = 3f_1 + 3f_2 \end{cases} ; \quad \begin{matrix} f_1(0) = 0 \\ f_2(0) = 1 \end{matrix}$

<u>Sol:</u> $\quad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}; \quad f' = \begin{pmatrix} f_1' \\ f_2' \end{pmatrix};$

$f' = Af; \quad A = \begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix} \Rightarrow$

$\mathcal{X}_A(x) = \det(xI_2 - A) = \det\begin{pmatrix} x-2 & -4 \\ -3 & x-3 \end{pmatrix} =$

$= (x-2)(x-3) - 12 = x^2 - 5x + 6 - 12 = x^2 - 5x - 6$

$= (x-6)(x+1) \Rightarrow \quad \lambda_1 = 0; \ \lambda_2 = -1$ (Different eigenvalues $\Rightarrow$ A diagonalizable !!)

<u>So:</u> for $\lambda_1 = 6$, solve $(6I_2 - A)x = 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$;

$\left(\begin{array}{cc|c} 4 & -4 & 0 \\ -3 & 3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \begin{matrix} x_2 = t; \ t \text{ scalar} \\ x_1 = t \end{matrix}$

$x = \begin{pmatrix} t \\ t \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}; \Rightarrow \quad X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix};$

for $\lambda_2 = -1$; Solve $((-1)I_2 - A)X = 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$;

$\left(\begin{array}{cc|c} -3 & -4 & 0 \\ -3 & -4 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} -3 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right); \quad \begin{matrix} x_2 = t; \ t \text{ scalar} \\ x_1 = \frac{4}{-3}t \end{matrix}$

$X = \begin{pmatrix} t\,4/-3 \\ t \end{pmatrix} = t\begin{pmatrix} -4/3 \\ 1 \end{pmatrix}; \quad X_2 = \begin{pmatrix} -4/3 \\ 1 \end{pmatrix}.$

So $f(x) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{6x} + c_2\begin{pmatrix} -4/3 \\ 1 \end{pmatrix}e^{-x}$. To get c_1, c_2:

Recall $\begin{cases} f_1(0)=0 \\ f_2(0)=1 \end{cases} \Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} =$

$$= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^0 + c_2 \begin{pmatrix} -4/3 \\ 1 \end{pmatrix} e^{-0} \Rightarrow$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} c_2(-4/3) \\ c_2 \end{pmatrix} \quad R$$

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -4/3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -4/3 & 0 \\ 1 & 1 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left(\begin{array}{cc|c} 1 & -4/3 & 0 \\ 0 & 7/3 & 1 \end{array}\right)$$

$$7/3\, c_2 = 1 \Rightarrow c_2 = 3/7 \ ; \qquad c_1 = 4/3 \cdot c_2 = \frac{4}{3} \cdot \frac{3}{7} = \frac{4}{7}$$

So

$$f(x) = \frac{4}{7} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{6x} + \frac{3}{7} \begin{pmatrix} -4/3 \\ 1 \end{pmatrix} e^{-x}$$

---

if Time: 1c/132