Lecture =THU=

§ 2.3 DIAGONALIZATION

§§ Eigenvalues and Eigenvectors

Def: If A is an $n\times n$ (square) mx., a number λ is called an eigenvalue of A if $AX=\lambda X$ for some $X\neq 0$.
Such a non-zero column X is called an eigenvector of A corresponding to the eigenvalue λ.

EXP: Given $A=\begin{pmatrix}7 & 3\\ 3 & -1\end{pmatrix}$, is $\lambda=-2$ an eigenvalue of A?

SOL: We need to solve $AX=\lambda X$; $AX=(-2)X$; $AX+2X=0$
$(A+2I)X=0$. So $\left[\begin{array}{cc|c}9 & 3 & 0\\ 3 & 1 & 0\end{array}\right] \xrightarrow{R_2-\frac{1}{3}R_1} \left(\begin{array}{cc|c}9 & 3 & 0\\ 0 & 0 & 0\end{array}\right) \xrightarrow{\frac{1}{9}R_1}$
$\left(\begin{array}{cc|c}1 & \frac{1}{3} & 0\\ 0 & 0 & 0\end{array}\right)$: ∞-ly many sol, so non-trivial solutions.

$x_2=t$ scalar; $x_1=-\frac{1}{3}t \Rightarrow X=\begin{pmatrix}-1/3\,t\\ t\end{pmatrix}=t\begin{pmatrix}-1/3\\ 1\end{pmatrix}$
choose $t=1 \Rightarrow X=\begin{pmatrix}-1/3\\ 1\end{pmatrix}$, it is NOT 0!!

EXP: is $\begin{pmatrix}1\\-2\\1\end{pmatrix}$ an eigenvector of $A=\begin{pmatrix}3 & 6 & 7\\ 3 & 3 & 7\\ 5 & 6 & 5\end{pmatrix}$? if YES, find eigenvalue.

SOL: $\underbrace{\begin{pmatrix}3 & 6 & 7\\ 3 & 3 & 7\\ 5 & 6 & 5\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}1\\-2\\1\end{pmatrix}}_{X}=-2\underbrace{\begin{pmatrix}1\\-2\\1\end{pmatrix}}_{X}=-2X$. Since $X\neq 0 \Rightarrow$

Answer is YES; $\lambda=-2$

Q: How do we get all eigenvalues?

Def The characteristic polynomial of an $n\times n$ mx. is $c_A(x) = \det(xI_n - A)$. Its degree is n

(THM) Let A be an $n\times n$ mx.

(1) The eigenvalues λ of A are the roots of the characteristic polynomial $c_A(x)$ of A.

(2) To get the eigenvectors X corresponding to λ, Just solve $(\lambda I - A)X = 0$.

2c/99 $A = \begin{pmatrix} 7 & 0 & 5 \\ 0 & 5 & 0 \\ -4 & 0 & -2 \end{pmatrix}$; find c.p., e, e.

Sol: $xI_3 - A = \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} - \begin{pmatrix} 7 & 0 & 5 \\ 0 & 5 & 0 \\ -4 & 0 & -2 \end{pmatrix} = \begin{pmatrix} x-7 & 0 & -5 \\ 0 & x-5 & 0 \\ 4 & 0 & x+2 \end{pmatrix}$

So $c_A(x) = (x-5)(-1)^{2+2} \det\begin{pmatrix} x-7 & -5 \\ 4 & x+2 \end{pmatrix} = (x-5)\left[(x-7)(x+2)+20\right]$

$= (x-5)(x^2 - 5x + 6) = (x-5)(x-2)(x-3)$.

Solve: $(x-5)(x-2)(x-3)=0 \Rightarrow x_1 = 5,\quad x_2 = 2,\quad x_3 = 3.$

$\boxed{\text{FOR } x_1 = 5}$ solve $(5I_3 - A)X = 0 \Rightarrow \left[\begin{array}{ccc|c} -2 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 7 & 0 \end{array}\right] \xrightarrow{R_2+2R_1}$

$\left[\begin{array}{ccc|c} -2 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & \frac{5}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $X = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$b = t$, t scalar $\Rightarrow c = 0,\ a = -\frac{5}{2}c = 0 \Rightarrow X = t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, t scalar

$\boxed{\text{For } \lambda_2 = 2}$ solve $(2I_3 - A)X = 0 \Rightarrow \left(\begin{array}{ccc|c} -5 & 0 & -5 & 0 \\ 0 & -3 & 0 & 0 \\ 4 & 0 & 4 & 0 \end{array}\right) \xrightarrow{\frac{1}{-5}R_1,\ \frac{1}{-3}R_2,\ \frac{1}{4}R_3}$

$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$. Say $X = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $c = t$, t scalar

$b = 0$; $a = -t$ $\Rightarrow$ $X = \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ t scalar

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$\boxed{\text{FOR } \lambda_3 = 3}$ Solve: $(3I_3 - A)X = 0 \Rightarrow \left(\begin{array}{ccc|c} -4 & 0 & -5 & 0 \\ 0 & -2 & 0 & 0 \\ 4 & 0 & 5 & 0 \end{array}\right) \xrightarrow{R_3 + R_1,\ \frac{1}{-2}R_2}$

$\left(\begin{array}{ccc|c} -4 & 0 & -5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow$ if $X = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow$ $c = t$, t scalar

$b = 0$

$a = -\frac{5}{4}t$

So $X = \begin{pmatrix} -\frac{5}{4}t \\ 0 \\ t \end{pmatrix} = t\begin{pmatrix} -5/4 \\ 0 \\ 1 \end{pmatrix}$; t scalar

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- One may get $<$ repeated roots ! / complex roots !.

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§§ DIAGONALIZATION

Def An $n \times n$ mx D is called a diagonal mx. if its off main diagonal entries are zero.

EXP: $D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$; $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$

DEF An $n \times n$ mx A is called diagonalizable if $P^{-1}AP = D$ for some invertible mx P and some diagonal mx. D.

$\boxed{\text{WHY?}}$ FACT: powers of such D, and then of A are easy to compute !

• If $D=\begin{pmatrix}5&0\\0&-1\end{pmatrix} \Rightarrow D^2 = D\cdot D = \begin{pmatrix}5&0\\0&-1\end{pmatrix}\begin{pmatrix}5&0\\0&-1\end{pmatrix} = \begin{pmatrix}5^2&0\\0&(-1)^2\end{pmatrix}$

$\ldots\ldots, D^m = \begin{pmatrix}5^m&0\\0&(-1)^m\end{pmatrix}.$

• If $P^{-1}AP = D \Rightarrow A = PDP^{-1} \Rightarrow A^m = \underbrace{PDP^{-1}\cdot PDP^{-1}\cdots PDP^{-1}}_{m \text{ times}}$

$= P\cdot D^m \cdot P^{-1}$
as above . . .

(THM) Let A be an $n\times n$ mx.

(1) A is diagonalizable $\Longleftrightarrow$ it has eigenvectors $X_1, X_2, \ldots, X_n$ such that $P = [X_1\ X_2 \ldots X_n]$ is invertible

(2) In this case: $P^{-1}AP = \begin{pmatrix}\lambda_1 & & & 0\\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n\end{pmatrix}$, where

each λ_i is the eigenvalue corresponding to X_i.

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before exercises: When is that possible?

THM: If an $n\times n$ mx. A has n distinct eigenvalues then A is diagonalizable

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2 e/93 $\quad A = \begin{pmatrix}1&1&0\\1&1&-1\\0&1&1\end{pmatrix}$

SOL: $xI_3 - A = \begin{pmatrix}x&0&0\\0&x&0\\0&0&x\end{pmatrix} - \begin{pmatrix}1&1&0\\1&1&-1\\0&1&1\end{pmatrix} = \begin{pmatrix}x-1&-1&0\\-1&x-1&1\\0&-1&x-1\end{pmatrix}$

$0 = \det\begin{pmatrix}x-1&-1&0\\-1&x-1&1\\0&-1&x-1\end{pmatrix} = (x-1)\det\begin{pmatrix}x-1&1\\-1&x-1\end{pmatrix} +$

$$+(-1)(-1)^{1+2} \det \begin{pmatrix} -1 & 1 \\ 0 & x-1 \end{pmatrix} = (x-1)\left[(x-1)^2+1\right]$$

$$+(-1)(x-1) = (x-1)\left[(x-1)^2+1-1\right]$$

$$= (x-1)^3 \Rightarrow \qquad x_1 = x_2 = x_3 = 1$$

<u>Find eigenvectors</u> for $x_1 = 1$

solve $(1 \cdot I_3 - A)X = 0 \Rightarrow \left(\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$X = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $\quad c = t$, t scalar
$\qquad b = 0 \qquad \Rightarrow X = t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow P = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} ?$
$\qquad a = t$

Even if we can construct $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, P does NOT exist!

so: not diagonalizable !!

The Algorithm for Diagonalization / pg 96

$\boxed{1}$ find the distinct eigenvalues of A

$\boxed{2}$ for each eigenvalue, find the basic solutions of $(xI - A)X = 0$

(3) A is diagonalizable $\iff$ there are n vectors found at $\boxed{2}$.

(4) Construct P with the vectors in $\boxed{2}$!!
Construct $D = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}$.

<u>Keep the ORDER!</u>

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§§§ <u>SIMILARITY</u>

<u>Def</u> A and B are similar if $B = P^{-1}AP$ for some inv. mat.

We write: $A \sim B$.

NOTE: $A \sim A$; $A \sim B \Leftrightarrow B \sim A$; $A \sim B, B \sim C \Rightarrow A \sim C$.

THM if A, B are similar, then

(1) $\det A = \det B$; (2) $C_A(x) = C_B(x)$; (3) A, B have the same eigenvalues.

Pf: @ blackboard since it's easy ---,

NOTE: $A \sim B \Rightarrow \begin{cases} A^{-1} \sim B^{-1} \\ A^T \sim B^T \\ A^k \sim B^k, \quad k \geq 0. \end{cases}$

show some of them.

12a/99 ---

6a/99 --- NOT ----

if time: 11/99