Lecture TUE

§ COMPOSITION

Def Let $T: \mathbb{R}^n \to \mathbb{R}^m$, $S: \mathbb{R}^m \to \mathbb{R}^k$ be 2 transformations. We define The COMPOSITE Transformation $S \circ T: \mathbb{R}^n \to \mathbb{R}^k$;

$S\circ T(x) = S(T(x))$; x in $\mathbb{R}^n$.

Picture:

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, bend left, "T"] \arrow[rr, bend right, dashed, "S\circ T"'] & \mathbb{R}^m \arrow[r, bend left, "S"] & \mathbb{R}^k \end{tikzcd}$$

THM: • $S\circ T$ is linear if S, T are linear transformations

•• if S, T have standard matrices A, B, then:

$S\circ T$ has standard mx: AB.

Pf: $S\circ T(x) = S(T(x)) = S(Bx) = A(Bx) = (AB)x$.

From here you get: linearity •

20/274 $T\circ T$ has standard mx: $A\cdot A = A^2 = A$.

Hence: $T\circ T(x) = Ax = T(x)$ for all x. So: $T\circ T = T$

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§ INVERSES

Def: Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a L.T. (SAME) A linear transformation $S: \mathbb{R}^n \to \mathbb{R}^n$ is called an inverse of T if $T(S(x)) = x$ and $S(T(x)) = x$ for all x in $\mathbb{R}^n$.

(in other words: $T\circ S = \mathbb{1}_{\mathbb{R}^n}$; $S\circ T = \mathbb{1}_{\mathbb{R}^n}$)

NOTE: The inverse (when exists) is unique!!

NOTATION: The inverse of a L.T. T is denoted by T^{-1}.

Def: T is called invertible if it has an inverse.

THM Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a L.T. with standard mx A. Then: (1) T has an inverse $\iff$ A is an inv. mx

(2) In this case A^{-1} is the standard mx. of T^{-1}.

Do the proof @ blackboard !!...!

186/274 $\mathbb{R}^n \xrightarrow{S} \mathbb{R}^n$, $T: \mathbb{R}^n \to \mathbb{R}^n$ both invertible. Show $S \circ T$ is invertible.

SOL Guess the "inverse": $T^{-1} \circ S^{-1} =: C$.

NOTE: * $(S \circ T) \circ C\,(X) = S\,(T(C(X))) =$

$= S(T(T^{-1}(S^{-1}(X)))) = S(S^{-1}(X)) = X.$

** $C \circ (S \circ T)\,(X) = C(S(T(X))) = T^{-1}(S^{-1}(S(T(X))))$

$= T^{-1}(T(X)) = X$; X in $\mathbb{R}^n$. So $S \circ T$ is invertible, AND $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$ (our C !!)

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§ Changing Coordinates

Lemma: if $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$ is a basis of $\mathbb{R}^n$, let $P = [F_1 \; F_2 \; \dots \; F_n]$. Then: $C_{\mathcal{F}}(X) = P^{-1} X$, X in $\mathbb{R}^n$,

where $C_{\mathcal{F}}(X) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$; $X = x_1 F_1 + x_2 F_2 + \dots + x_n F_n$, is called the $\mathcal{F}$-coordinate vector of X.

In particular $C_{\mathcal{F}}: \mathbb{R}^n \to \mathbb{R}^n$ is linear with standard mx. P^{-1}).

Pf: $P \cdot C_{\mathcal{F}}(X) = P \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [F_1 \; F_2 \dots F_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} =$

$= x_1 F_1 + x_2 F_2 + \cdots + x_n F_n = X$. Since $\mathcal{F}$ is a basis $\Rightarrow$ P is invertible (so P^{-1} exists) $\Rightarrow$ $C_{\mathcal{F}}(X) = P^{-1}X$.

Def $C_{\mathcal{F}} : \mathbb{R}^n \to \mathbb{R}^n$ is called the coordinate operator (l.t.).

- What if we have 2 bases? $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$ and $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ basis of $\mathbb{R}^n$. HOW ARE $C_{\mathcal{F}}(X)$ and $C_{\mathcal{G}}(X)$ related?

A: Recall: $C_{\mathcal{F}}(X) = P^{-1}X$, $C_{\mathcal{G}}(X) = Q^{-1}X$, $Q = [G_1 \cdots G_n]$.

So: $C_{\mathcal{G}}(X) = Q^{-1}X = Q^{-1} P\, C_{\mathcal{F}}(X) = Q^{-1} [F_1\ F_2 \cdots F_n] \cdot C_{\mathcal{F}}(X) = [Q^{-1}F_1\ \ Q^{-1}F_2 \cdots Q^{-1}F_n] \cdot C_{\mathcal{F}}(X) =$

$$= \underbrace{\begin{bmatrix} C_{\mathcal{G}}(F_1) & C_{\mathcal{G}}(F_2) & \cdots & C_{\mathcal{G}}(F_n) \end{bmatrix}}_{\text{mx: } P_{\mathcal{G} \leftarrow \mathcal{F}}} C_{\mathcal{F}}(X)$$

, called "THE CHANGE Mx" from $\mathcal{F}$ to $\mathcal{G}$.

So:

(THM) $C_{\mathcal{G}}(X) = P_{\mathcal{G} \leftarrow \mathcal{F}} \cdot C_{\mathcal{F}}(X)$

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15/274 $\quad \mathcal{F} = \left\{ \begin{pmatrix}1\\1\end{pmatrix}; \begin{pmatrix}0\\1\end{pmatrix} \right\}$; $G = \left\{ \begin{pmatrix}2\\3\end{pmatrix}; \begin{pmatrix}1\\2\end{pmatrix} \right\}$

- $P_{G \leftarrow F} = ?$ $\quad P_{F \leftarrow G} = ?$; Show $C_G(x) = P_{G \leftarrow F} C_{\mathcal{F}}(x)$

Sol: • $C_{\mathcal{F}}(x) = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}^{-1} X = \frac{1}{1-0}\begin{pmatrix}1 & 0\\-1 & 1\end{pmatrix} X =$

$= \begin{pmatrix}1 & 0\\-1 & 1\end{pmatrix} X$; If $X = \begin{pmatrix}x_1\\x_2\end{pmatrix} \Rightarrow C_{\mathcal{F}}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1 & 0\\-1 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$

$= \begin{pmatrix}x_1\\-x_1+x_2\end{pmatrix}$.

• $C_G(x) = \begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}^{-1} X = \frac{1}{4-3}\begin{pmatrix}2 & -1\\-3 & 2\end{pmatrix} X =$

$= \begin{pmatrix}2 & -1\\-3 & 2\end{pmatrix} X$. If $X = \begin{pmatrix}x_1\\x_2\end{pmatrix} \Rightarrow C_G\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}2 & -1\\-3 & 2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$

$= \begin{pmatrix}2x_1 - x_2\\-3x_1 + 2x_2\end{pmatrix}$.

• $P_{G \leftarrow F} = \left[C_G(F_1) \quad C_G(F_2) \right] =$

$= \begin{bmatrix}1 & -1\\-1 & 2\end{bmatrix}$

• $P_{\mathcal{F} \leftarrow G} = \left[C_{\mathcal{F}}(G_1) \quad C_{\mathcal{F}}(G_2) \right] =$

$= \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}$

NOTE:

$\begin{pmatrix}1 & -1\\-1 & 2\end{pmatrix}^{-1} = \frac{1}{2-1}\begin{pmatrix}2 & 1\\1 & 1\end{pmatrix} = \begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}$

Check: $C_G(x) = P_{G \leftarrow F} \; C_F(x)$

$$RS = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ -x_1 + x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_1 - x_2 \\ -x_1 - 2x_1 + 2x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1 - x_2 \\ -3x_1 + 2x_2 \end{pmatrix} = L.S.$$

$m < n$

4/274 $\quad T : \mathbb{R}^n \to \mathbb{R}^m \quad T \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$

SOL: EASY $\to$ linearity!!

standard mx. $A = \left[T(E_1) \;\; \cdots \;\; T(E_n) \right]$

$$= \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ m \to \end{array} \begin{bmatrix} \underbrace{\begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & & \vdots \\ 0 & 0 & & \vdots \\ 0 & \vdots & & \vdots \\ 0 & \vdots & & \vdots \\ 0 & \vdots & & 1 \\ 0 & 0 & & \end{matrix}}_{I_m} & \begin{matrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ 0 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \vdots & 0 \\ & & & \end{matrix} \end{bmatrix} = [I_m \mid O]$$