Lecture THU

PROBLEMS (5.3) 3a/319 $T(z_1+z_2) = \overline{z_1+z_2} = \overline{z_1} + \overline{z_2} = T(z_1) + T(z_2)$.

ALSO: $T(rz) = \overline{rz} = \overline{r}\,\overline{z} = r\overline{z}$ (since $r = \overline{r}$ for any real # r). Note $rT(z) = r\overline{z}$, so $T(rz) = rT(z)$.

$\boxed{3c/319}$ $\boxed{T(p+q) = T(p) + T(q)}$ Suppose $p = a_0 + a_1x + \cdots + a_nx^n$, $q = b_0 + b_1x + \cdots + b_nx^n$; Then $p+q = (a_0+b_0) + (a_1+b_1)x + \cdots + (a_n+b_n)x^n$

Note: $T(p+q) = a_n + b_n \overset{!}{=} T(p) + T(q)$. Next: $T(rp) = T(r(a_0 + a_1x + \cdots + a_nx^n)) = T(ra_0 + ra_1x + \cdots + ra_nx^n) = ra_n = rT(p)$, since $T(p) = a_n$.

4a/319 $T: M_{n,n} \to \mathbb{R}$, $T(A) = \det A$.

SOL: if it were linear $\Rightarrow$ $T\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}\right) = T\left(\begin{pmatrix}1&0\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\0&1\end{pmatrix}\right) \overset{!}{=} T\begin{pmatrix}1&0\\0&0\end{pmatrix} + T\begin{pmatrix}0&0\\0&1\end{pmatrix} = 0 + 0 = 0$. So $\det\begin{pmatrix}1&0\\0&1\end{pmatrix} = 0 \Rightarrow 1 = 0$, contradiction. So, it is NOT linear.

1f/319 LOOK AT: $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x) = 0$, x in $\mathbb{R}^2$

NOTE: $\mathbb{R}^2 \neq \{0\}$. so (F) !!!

1g/319 Let v be in V. Then $T(v)$ is in $\operatorname{im} T = \{0\}$. So $T(v) = 0$. Since v is arbitrary taken from $V \Rightarrow T = 0$; (Recall $0(v) = 0$ for all v in V).

so: True

$\boxed{1i}$/319 FALSE; look at $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x) = x$ for all x in $\mathbb{R}^2$

IT IS $\left\langle \begin{array}{l}\text{ONTO}\\ \text{1-1}\end{array}\right.$;

1j/319 1-1 $\Rightarrow$ $\dim \operatorname{Ker} T = 0 \Rightarrow \dim V = \dim \operatorname{Im} T$

$\Rightarrow \dim W = \dim \operatorname{Im} T.$

- But $\operatorname{Im} T \subseteq W$, and $\dim W$ is finite $\Bigg\} \Rightarrow \boxed{\operatorname{Im} T = W}$

So: True

1K/319: F look at: $T: \mathbb{R}^6 \to \mathbb{R}^5$, $T(x) = 0$ for all x in $\mathbb{R}^6$. $\operatorname{Im} T = \{0\} \neq \mathbb{R}^5$; T is linear.

2g/319 $\boxed{T \text{ is ONTO;}} \overset{?}{\Rightarrow} \boxed{\dim V \geq \dim W}$

We assume V is FINITE dim. Then (Dim. Thm)

$\dim V = \dim \underbrace{\operatorname{Im} T}_{W} + \dim \operatorname{Ker} T = \dim W + \dim \operatorname{Ker} T$

- $\geq \dim W + 0 = \dim W.$

2h/319 look AT: $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$

is is LINEAR, $3 \geq 2$, but NOT onto. so F

§ 4.9 LINEAR Transformations

Recall: every $m \times n$ mx A gives rise to a L.T. $T: \mathbb{R}^n \to \mathbb{R}^m$, $T(X) = AX$. The converse:

THM Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear t. Then there is an $m \times n$ mx. A such that: $T = S$, where $S: \mathbb{R}^n \to \mathbb{R}^m$; $S(X) = AX$.

- Pf: Let $A = [T(E_1)\ T(E_2) \cdots T(E_n)]$, where $I_n = [E_1\ E_2 \cdots E_n]$. Let $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ in $\mathbb{R}^n$. Note:

$$T(X) = T(x_1 E_1 + x_2 E_2 + \cdots + x_n E_n) \overset{\text{lin.}}{=} x_1 T(E_1) + x_2 T(E_2) + \cdots + x_n T(E_n) = \begin{bmatrix} T(E_1) & T(E_2) & \cdots & T(E_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= AX = S(X);$$

DEF: A is called " The standard matrix of T ".

6a/274 $T: \mathbb{R}^3 \to \mathbb{R}^3$; $T\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}3\\-1\\3\end{pmatrix}$; $T\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}2\\1\\3\end{pmatrix}$; $T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\-2\\0\end{pmatrix}$. We need to find $T(X) = T\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$.

SOL: STEP 1 find the standard mx. of T:

$$E_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} = a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\1\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix} \Rightarrow \begin{cases} a = 1 \\ a+b = 0 \\ b+c = 0 \end{cases} \Rightarrow \begin{cases} a=1 \\ b=-1 \\ c=1 \end{cases}$$

So $T(E_1) = T\left(1\begin{pmatrix}1\\1\\0\end{pmatrix} + (-1)\begin{pmatrix}0\\1\\1\end{pmatrix} + 1\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = 1\begin{pmatrix}3\\-1\\3\end{pmatrix} + (-1)\begin{pmatrix}2\\1\\3\end{pmatrix} + 1\begin{pmatrix}-1\\-2\\0\end{pmatrix} = \begin{pmatrix}0\\-4\\0\end{pmatrix}$.

$$E_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} = a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\1\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix} \Rightarrow \begin{cases} a = 0 \\ a+b = 1 \\ b+c = 0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=1 \\ C=-1 \end{cases}$$

So $T(E_2) = T\left(0\begin{pmatrix}1\\1\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\\1\end{pmatrix} + (-1)\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = 0\begin{pmatrix}3\\-1\\3\end{pmatrix} + 1\begin{pmatrix}2\\1\\3\end{pmatrix} + (-1)\begin{pmatrix}-1\\-2\\0\end{pmatrix} = \begin{pmatrix}3\\3\\3\end{pmatrix}$.

$$E_3 = \begin{pmatrix}0\\0\\1\end{pmatrix} = a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\1\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix} \Rightarrow \begin{cases} a = 0 \\ a+b = 0 \\ b+c = 1 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0 \\ c=1 \end{cases}$$

So: $T(E_3) = \left(0\begin{pmatrix}1\\1\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\\1\end{pmatrix} + 1\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = 0\begin{pmatrix}3\\-1\\3\end{pmatrix} + 0\begin{pmatrix}2\\1\\3\end{pmatrix} + 1\begin{pmatrix}-1\\-2\\0\end{pmatrix} = \begin{pmatrix}-1\\-2\\0\end{pmatrix}$.

HENCE $A = \begin{bmatrix} T(E_1) & T(E_2) & T(E_3) \end{bmatrix} = \begin{bmatrix} 0 & 3 & -1 \\ -4 & 3 & -2 \\ 0 & 3 & 0 \end{bmatrix}$

Hence: $T(X) \overset{!}{=} AX = \begin{bmatrix} 0 & 3 & -1 \\ -4 & 3 & -2 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 - x_3 \\ -4x_1 + 3x_2 - 2x_3 \\ 3x_3 \end{bmatrix}$

§ COMPOSITION

Def Let $T: \mathbb{R}^n \to \mathbb{R}^m$, $S: \mathbb{R}^m \to \mathbb{R}^k$ be 2 transformations. We define the <u>COMPOSITE</u> <u>Transformation</u>

$$S \circ T: \mathbb{R}^n \longrightarrow \mathbb{R}^m, \quad S \circ T(X) = S(T(X)), \; X \text{ in } \mathbb{R}^n$$

PICTURE:

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, bend left, "T"] \arrow[rr, dashed, bend right] & \mathbb{R}^m \arrow[r, "S"'] & \mathbb{R}^k \end{tikzcd}$$

THM: • $S \circ T$ is linear if S, T ($S \circ T$) are L.T.

•• if S, T have standard matrices A, B, then $S \circ T$ has standard mx: AB.

Pf: (easy!!) $S \circ T(X) = S(T(X)) = S(BX) = A(BX)$
$= (AB)X$; (from here we may see: <u>linearity</u>!!)

20/274. $T \circ T$ has standard mx: $A \cdot A = A^2 (= A)$
hence $T \circ T(x) = AX = T(X)$ for all X. So
$T \circ T = T$

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§ INVERSES

Def: Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a L.T. A L.T.
$S: \mathbb{R}^n \to \mathbb{R}^n$ is called <u>an</u> inverse of T if:

$$T(S(x)) = X \quad \underline{\text{and}} \quad S(T(x)) = X; \text{ all } x \text{ in } \mathbb{R}^n.$$

SAME!!

NOTE: The inverse (when exists) <u>is</u> unique!!

• NOTATION The inverse of a L.T. T that has inverse is called/denoted T^{-1}.
(T is said to be invertible!!)

$\boxed{\text{THM}}$ Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a L.T. with standard mx A. Then:

① T has an inverse $\iff$ A is inv. mx.

② In this case A^{-1} is the standard mx. of T^{-1}.

18b/274 $\mathbb{R}^n \xrightarrow{S} \mathbb{R}^n$, $T: \mathbb{R}^n \to \mathbb{R}^n$ both inv.

show $S \circ T$ is inv.

SOL: Guess the "inverse": $T^{-1} \circ S^{-1} =: C$

NOTE: $(S \circ T) \circ C(X) = S(T(C(X))) = S(T(T^{-1}(S^{-1}(X)))) = S(S^{-1}(X)) = X$; $C \circ (S \circ T)(X) =$
$= (T^{-1} \circ S^{-1}) \circ (S \circ T)(X) = T^{-1}(S^{-1}(S(T(X)))) = T^{-1}(T(X)) = X$; X in $\mathbb{R}^n$! So $S \circ T$ is inv.
and $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$ (our $\underline{\underline{C}}$)!
