Lecture TUE : 17 March (5.3

THM Let $T: V \to W$, $S: V \to W$ be 2 linear transformations. Assume that $V = \text{span}\{v_1, \dots, v_k\}$. If $S(v_i) = T(v_i)$ for each i, then $S = T$

SOL: $v = r_1 v_1 + \dots + r_k v_k \Rightarrow S(v) = S(r_1 v_1 + \dots + r_k v_k) =$
$= r_1 S(v_1) + \dots + r_k S(v_k) \overset{!}{=} r_1 T(v_1) + \dots + r_k T(v_k) = T(r_1 v_1 + \dots + r_k v_k)$
$= T(v)$: Done

Lemma: if $\{b_1, b_2, \dots, b_n\}$ is a basis of V, every vector v in V can be uniquely represented as a linear combo: $v = r_1 b_1 + \dots + r_n b$

Pf: If $v = s_1 b_1 + \dots + s_n b_n$ is another representation, then $(s_1 - r_1) b_1 + \dots + (s_n - r_n) b_n = 0 \overset{L.I.}{\Rightarrow} s_1 = r_1; \dots, s_n = r_n$.

THM Let $\{b_1, \dots, b_n\}$ be a basis of V. Let $\{w_1, w_2, \dots, w_n\}$ be some vectors in W. There is a unique linear transformation $T: V \to W$ such that: $T(b_i) = w_i$ for all i.

Pf: Existence: $T(r_1 b_1 + \dots + r_n b_n) \overset{D}{=} r_1 w_1 + \dots + r_n w_n$.
Uniqueness: from Thm above

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Q: is $\{1, x+x^2, x-x^2\}$ a basis?

$a \cdot 1 + b(x+x^2) + c(x-x^2) = 0 \Rightarrow \begin{cases} a = 0 \\ b+c=0 \\ b-c=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0 \\ c=0 \end{cases}$ L.I.

So a basis.

Note: $v = a + bx + cx^2 = a \cdot 1 + b\left(\frac{x+x^2}{2} + \frac{x-x^2}{2}\right) +$
$+ c\left(\frac{x+x^2}{2} - \frac{x-x^2}{2}\right) = a \cdot 1 + \frac{b+c}{2}(x+x^2) + \left(\frac{b-c}{2}\right)(x-x^2)$

So: $T(v) = a\cdot x^4 + \frac{b+c}{2}\cdot 1 + \left(\frac{b}{2} - \frac{c}{2}\right)(x + x^3)$.

~~Do THEN:~~ $\Rightarrow$ $\boxed{1\ a,b\ /319}$

§ KERNEL And IMAGE

Any $T: V \to W$ linear, gives rise to 2 subspaces:

Kernel of T: $\quad \operatorname{Ker} T = \{ v \in V \mid T(v) = 0 \} \subseteq V$;

Image of T: $\quad \operatorname{Im} T = \{ T(v) \mid v \text{ in } V \} \subseteq W$.

Lemma: $\operatorname{Ker} T$, $\operatorname{Im} T$ are subspaces in V, W.

Pf: Easy.

EXP: A $m\times n$ mx, $T: \mathbb{R}^n \to \mathbb{R}^m$, $T(X) = AX$. Then $\operatorname{Ker} T = \{ X \mid AX = 0 \} = \operatorname{null} A$ and $\operatorname{Im} T = \{ T(X) \mid X \text{ in } \mathbb{R}^n \} \overset{!}{=} \{ AX \mid X \text{ in } \mathbb{R}^n \} \overset{!}{=} \operatorname{im} A$.

EXP: $T: M_{n,n} \to M_{n,n}$, $T(A) = A - A^T$. It is linear. Find $\operatorname{Ker} T$, $\operatorname{Im} T$.

SOL: $\operatorname{Ker} T = \{ A \mid A = A^T \} = \{ A \mid A \text{ is symmetric} \}$

$\operatorname{Im} T = \{ S \text{ in } M_{n,n} \mid T(A) = S \text{ for some } A \text{ in } M_{n,n} \} =$

$= \{ S \mid S = A - A^T \text{ for some } A \text{ in } M_{n,n} \} =$

$\overset{!!}{=} \{ S \mid S = -S^T \}$

DEF: $T: \cancel{V} \to W$ is ONTO if $\operatorname{Im} T = W$

$T: V \to W$ is one-to-one if

$T(v_1) = T(v_2) \Rightarrow v_1 = v_2$

EXP: $T: \mathbb{R}^2 \to \mathbb{R}$, $T\begin{pmatrix} x \\ y \end{pmatrix} = y$ is ONTO, NOT 1-1
$T: \mathbb{R} \to \mathbb{R}^2$, $T(x) = \begin{pmatrix} x \\ 0 \end{pmatrix}$ is 1-1, NOT onto.

THM: Let $T: V \to W$ be linear transformation.

T is 1-1 $\iff$ $\operatorname{Ker} T = \{0\}$

Pf: blackboard!

(EXP) A is $m \times n$; $T: \mathbb{R}^n \to \mathbb{R}^m$, $T(X) = AX$.

CLAIM T is 1-1 $\iff$ rank $A = n$

SOL: rank $A = n$ $\iff$ $AX = 0$ has only the trivial solution; $\iff$ $T(X) = 0$ has only the trivial solution $\iff$ $\operatorname{Ker} T = \{0\}$ $\iff$ T is 1-1.

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§ The DIMENSION Theorem Let $T: V \to W$ be a linear transformation. if both $\operatorname{Ker} T$ and $\operatorname{im} T$ are finite dimensional, then V is also finite dimensional, and:

$$\boxed{\dim V = \dim(\operatorname{im} T) + \dim(\operatorname{Ker} T)}$$

COR Let $T: V \to W$ be a linear transformation. if $\dim V = \dim W$ is finite, then T is 1-1 $\iff$ T is onto.

PROBLEMS:

3a/319 $T(z_1+z_2)=\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}=$

$=T(z_1)+T(z_2)$; $\begin{cases} T(rz)=\overline{rz}=\bar r\bar z \\ rT(z)=r\bar z \overset{!}{=} \bar r \bar z \end{cases}$? since $r=\bar r$ for any r in $\mathbb{R}$

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$T(p+q)\overset{?}{=}T(p)+T(q)$. Suppose $p=a_0+a_1x+\dots+a_nx^n$, $q=b_0+b_1x+\dots+b_nx^n$.

Then $p+q=(a_0+b_0)+(a_1+b_1)x+\dots+(a_n+b_n)x^n \Rightarrow$

$T(p+q)=(a_n+b_n)=a_n+b_n=T(p)+T(q)$

$T(rp)=T(r(a_0+a_1x+\dots+a_nx^n))=T(ra_0+ra_1x+\dots+ra_nx^n)=ra_n=r\cdot T(p)$ since $T(p)=a_n$.

4a/319 $T: M_{n,n}\to\mathbb{R}$ $T(A)=\det A$. NOT!

SOL: if it were $\Rightarrow$ $\underbrace{T\begin{pmatrix}1&0\\0&1\end{pmatrix}}_{=1}=T\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\0&1\end{pmatrix}\right)=\underbrace{T\begin{pmatrix}1&0\\0&0\end{pmatrix}}_{=0}+\underbrace{T\begin{pmatrix}0&0\\0&1\end{pmatrix}}_{=0}=0$ contradic!

1f/319 (F) $T:\mathbb{R}^2\to\mathbb{R}^2$ $T(x)=0$ all x in $\mathbb{R}^2$

Note: $\mathbb{R}^2\neq\{0\}$

! $\dim V<\infty$ | if $\dim V=\infty$, it's easier: By Def! | a blackboard....

1g/319 (T) $\operatorname{Im} T=\{0\}\Rightarrow \dim V=\dim\ker T$ so done.

$\ker T\subseteq V \Rightarrow \ker T=V \Rightarrow$ "for every v in V $\Rightarrow$ $T(v)=0$"

1i/319 (F) $T:\mathbb{R}^2\to\mathbb{R}^2$, $T(x)=x$ < 1-1, onto

$2=2$!!

1j/319 1-1 $\Rightarrow$ $\dim\ker T=0 \Rightarrow \dim V=\dim\operatorname{Im}T$

$\left.\begin{array}{l}\Rightarrow \dim W=\dim \operatorname{Im}T \\ \text{but } \operatorname{Im}T\subseteq W\ (<\infty)!!\end{array}\right\}\Rightarrow \operatorname{Im}T=W.$

1.k/319 F $T: \mathbb{R}^0 \to \mathbb{R}^5$ $T(x)=0$ for all x

$\operatorname{Im} T = \{0\} \neq \mathbb{R}^5$, T is linear.

2g/319 $\boxed{T}$ T is onto $\Rightarrow$ $\operatorname{Im} T = W$

The dimension Theorem: $\dim V = \dim \operatorname{Im} T + \dim \operatorname{Ker} T = \dim W + \dim \operatorname{Ker} T \geq \dim W + 0 = \dim W$

HERE: in 2g we assume V is f-d.

2h/319 F

$T: \mathbb{R}^3 \to \mathbb{R}^2$ $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$

NOTE: $3 \geq 2$, but T is NOT onto

§ 4.9 Linear Transformations

Recall that every A an $m \times n$ mx gives rise to a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$, $T(X) = AX$.

The converse:

THM Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there is a mx A of size $m \times n$ such that $T = S$ where $S: \mathbb{R}^n \to \mathbb{R}^m$ is given by $S(X) = AX$.

Pf: Let $A = \left[T(E_1) \ \ldots \ T(E_n) \right]$ where $I_n = [E_1 \ldots E_n]$

Then $T(X) = T(x_1 E_1 + \ldots + x_n E_n) \overset{\text{lin.}}{=} x_1 T(E_1) + \ldots + x_n T(E_n)$

X in $\mathbb{R}^n$

$= A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = AX$, where $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

$\underset{SX}{\parallel}$

Def A is called "The standard matrix of T".

- 6 p/ 274 $\quad T: \mathbb{R}^3 \to \mathbb{R}^3$

$$T\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}3\\-1\\3\end{pmatrix}; \quad T\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}2\\1\\3\end{pmatrix}; \quad T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\-2\\0\end{pmatrix}$$

We need to find $T(X) = T\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$.

Sol: Step 1 Find the standard mx. of T:

$$E_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} = a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\1\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix} \Rightarrow$$

$$\begin{cases} a = 1 \\ a + b = 0 \\ b + c = 0 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = -1 \\ c = 1 \end{cases}$$

So $T(E_1) = T\left(1\begin{pmatrix}1\\1\\0\end{pmatrix} + (-1)\begin{pmatrix}0\\1\\1\end{pmatrix} + 1\begin{pmatrix}0\\0\\1\end{pmatrix}\right) =$

- $= 1\begin{pmatrix}3\\-1\\3\end{pmatrix} + (-1)\begin{pmatrix}2\\1\\3\end{pmatrix} + 1\begin{pmatrix}-1\\-2\\0\end{pmatrix} = \begin{pmatrix}3\\-1\\3\end{pmatrix} + \begin{pmatrix}-4\\-2\\-3\end{pmatrix} + \begin{pmatrix}-1\\-2\\0\end{pmatrix}$

$$= \begin{pmatrix}-2\\-5\\0\end{pmatrix}.$$

$$E_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} = a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\1\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix} \Rightarrow \begin{cases} a = 0 \\ a + b = 1 \\ b + c = 0 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = 1 \\ c = -1 \end{cases}$$

$$\Rightarrow T(E_2) = T\left(0\begin{pmatrix}1\\1\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\\1\end{pmatrix} + (-1)\begin{pmatrix}0\\0\\1\end{pmatrix}\right) =$$

$$= 0\begin{pmatrix}3\\-1\\3\end{pmatrix} + 1\begin{pmatrix}2\\1\\3\end{pmatrix} + (-1)\begin{pmatrix}-1\\-2\\0\end{pmatrix} = \begin{pmatrix}3\\3\\3\end{pmatrix}.$$

$$E_3 = \begin{pmatrix}0\\0\\1\end{pmatrix} = a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\1\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix} \Rightarrow \begin{cases} a = 0 \\ a + b = 0 \\ b + c = 1 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = 0 \\ c = 1 \end{cases}$$

- $T(E_3) = T\left(0\begin{pmatrix}1\\1\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\\1\end{pmatrix} + 1\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = 0\begin{pmatrix}3\\-1\\3\end{pmatrix} + 0\begin{pmatrix}2\\1\\3\end{pmatrix} + 1\begin{pmatrix}0\\0\\1\end{pmatrix}$

$$= \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Hence $A = [T(E_1) \quad T(E_2) \quad T(E_3)] =$

$$= \begin{bmatrix} -2 & 3 & 0 \\ -5 & 3 & 0 \\ 0 & 3 & 1 \end{bmatrix}. \quad \text{Hence } T(X) \overset{!}{=} AX$$

$$= \begin{pmatrix} -2 & 3 & 0 \\ -5 & 3 & 0 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_1 + 3x_2 \\ -5x_1 + 3x_2 \\ 3x_2 + x_3 \end{pmatrix} \text{ for any } X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

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§ Composition (D) Let $T: \mathbb{R}^n \to \mathbb{R}^m$, $S: \mathbb{R}^m \to \mathbb{R}^k$ be 2 linear transformations. We define: the COMPOSITE transformation by:

$S \circ T : \mathbb{R}^n \to \mathbb{R}^k$, $(S \circ T)(X) = S(T(X))$ for all X in $\mathbb{R}^n$.

THM
- $S \circ T$ is linear.
- If S, T have standard matrices A, B, then $S \circ T$ has standard mx: AB.

Pf: easy: $(S \circ T)(X) = S(T(X)) = S(BX) = A(BX) = (AB)X$ (from here we may see: linearity!!)

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$T \circ T(X) = T(T(X)) = T(AX) = A(AX)$
$= A^2 X = AX = T(X) \Rightarrow \boxed{T \circ T = T}$

OR: $T \circ T(X) \underset{\text{Thm above}}{=} A^2 X = AX = T(X) \Rightarrow T \circ T = T$