Lecture

Lemma (Independent Lemma) Let $\{v_1, v_2, ---, v_n\}$ be a L.i. subset of V; suppose v is not in span$\{v_1, --, v_n\}$ Then the larger set $\{v, v_1, v_2, ---, v_n\}$ is L.I.
(same proof as in $\mathbb{R}^n$)

Lemma (Dependent Lemma) A set of elements $\{v_1, v_2, ---, v_n\}$ is Linearly Dependent $\iff$ one of the v_i is a linear combo. of the others.
(Same proof!)

OF Course: A set of vectors (elements) is L.D. if it is not L.i (by Definition)

THM: Let $V \neq \{0\}$ be a finite dimensional vector space, and assume that V is spanned by n vectors

① V has a basis, and $\dim V \leq n$.

② Every lin. Ind. subset of V is a part of a basis.

③ Every finite spanning set of V contains a basis.

THM: Suppose that $\dim V = n$. Let B be a set of n vectors in V. Then: B is L.i $\iff$ B spans V. In either case: B is a basis of V.

THM: Let U be a subspace of a vector space V. Suppose $\dim V = n$.

(1) U has a basis, and $\dim U \le n$.

(2) if $\dim U = n$, then $U = V$.

(3) Every basis of U is a part of a basis of V

EXC: 3a/309 $U \subseteq V = \mathbb{P}_2$. Since $\dim P_2 = 3 \Rightarrow \dim U \le$

L.I.?

$a(1-2x) + b(3x - x^2) + c(3-2x^2) = 0$ ($\Leftarrow$ the zero polynomial). We are led to:

$$\begin{cases} a + 3c = 0 \\ -2a + 3b = 0 \\ -b - 2c = 0 \end{cases} \Rightarrow \left\{ \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \right\} \xrightarrow{R_2 + 2R_1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \xrightarrow{R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{L.D.}$$

In fact $c = t$, $t \#$, $b = -2t$, $a = -3t$. Chose $t = 1 \Rightarrow$

$3 - 2x^2 = 3(1-2x) + 2(3x - x^2) \overset{!}{\Rightarrow} U \overset{!!}{=} \text{span}\{1-2x, 3x-x^2\}$

$\boxed{\text{L.I.?}}$ $a(1-2x) + b(3x - x^2) = 0 \Rightarrow \left[\begin{array}{cc|c} a & 0 & 0 \\ -2a & 3b & 0 \\ 0 & -b & 0 \end{array}\right] \Rightarrow \begin{array}{l} a = 0 \\ b = 0 \end{array} \Rightarrow \underline{\text{L.I.}}$

So a basis is $\{1-2x, 3x-x^2\}$, $\Rightarrow \dim U = 2$

3c/309 $p(x) = p(-x) \Rightarrow a + bx + cx^2 = a - bx + cx^2 \Leftrightarrow b = 0$

$\Rightarrow U = \{p(x) \mid p(x) = a + cx^2\} = \text{span}\{1, x^2\}$.

Since $\{1, x^2\}$ is L.I. $\Rightarrow \{1, x^2\}$ is a basis.

$\overline{\uparrow}$ easy

DO/TRY: 3i/309, 11/309, 17/309, 21/310; 3/309; 24 a, b/3...

§5.3 LINEAR Transformations

Def • Let V, W be 2 vector spaces. A transformation $T: V \to W$ is a rule that associates with every vector v in V a unique vector $T(v)$ in W.

• A transformation $T: V \to W$ is called linear if:

$$\begin{cases} T(v_1+v_2) = T(v_1) + T(v_2) & \text{for all } v_1, v_2 \text{ in } V \\ T(rv) = rT(v) & \text{for all } r \ \#, \ v \text{ in } V. \end{cases}$$

(i.e. preserves $+, \cdot$)

V is called Domain, W is called Codomain.

EXP. Let A be an $m \times n$ mx. We may define $T: \mathbb{R}^n \to \mathbb{R}^m$ by $T(X) = AX$, X in $\mathbb{R}^n$. If? easy... DO IT!!

EXP: $T: M_{m,n} \to M_{n,m}$, $T(A) = A^T$: DO IT!

EXP:

$$\begin{cases} 1_V: V \to V, & 1_V(v) = v & \text{for all } v \text{ in } V. \\ 0: V \to W, & 0(v) = 0 & \text{for all } v \text{ in } V. \\ S_a: V \to V, & S_a(v) = av, & \text{for all } v \text{ in } V. \end{cases}$$

$a\#$

(THM) (Properties): Let $T: V \to W$ be a linear transformation

(1) $T(0) = 0$ (2) $T(-v) = -T(v)$ for all v in V

(3) $T(r_1 v_1 + \dots + r_k v_k) = r_1 T(v_1) + \dots + r_k T(v_k)$ for all $r_1, \dots, r_k$ #s, $v_1, \dots, v_k$ in V.

THM: Let $T: V \to W$, $S: V \to W$ be 2 linear transformati[ons]. Assume $V = \text{span}\{v_1, \dots, v_k\}$. If $S(v_i) = T(v_i)$ for eac[h i] then $S = T$.

SOL: $v = r_1 v_1 + \dots + r_k v_k \to$ Apply S, use $S(v_i) = T(v_i)$

$\Rightarrow$ Done.

Lemma: If $\{b_1, \dots, b_n\}$ is a basis of V, every vector v in V can be UNiquely represented as a linear combo: $v = r_1 b_1 + \dots + r_n b_n$.

Pf: If $v = s_1 b_1 + \dots + s_n b_n$ is another representation then $(s_1 - r_1) b_1 + \dots + (s_n - r_n) b_n = 0 \xRightarrow{L.I.} s_1 = r_1, \dots, s_n = r_n$

THM: Let $\{b_1, \dots, b_n\}$ be a basis of V. Let $\{w_1, \dots, w_n\}$ be some vectors in W. There is a unique linear transformation $T: V \to W$ such that: $T(b_i) = w_i$ for all i

Pf: $T(r_1 b_1 + \dots + r_n b_n) = r_1 w_1 + \dots + r_n w_n$; $r_1, \dots, r_n$ #

5 a/319 a is: $\{1, x+x^2, x-x^2\}$ a basis?

1 a, b/319

$a 1 + b(x+x^2) + c(x-x^2) = 0 \Rightarrow \begin{cases} a = 0 \\ b+c = 0 \\ b-c = 0 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = 0 \\ c = 0 \end{cases}$ L.I.

$$v = a + bx + cx^2 = a \cdot 1 + b\left(\frac{x+x^2+x-x^2}{2}\right) + c \cdot \left(\frac{x+x^2-(x-x^2)}{2}\right)$$

$$= a \cdot 1 + \left(\frac{b+c}{2}\right)(x+x^2) + \left(\frac{b}{2} - \frac{c}{2}\right)(x-x^2)$$

$$T(v) = a \cdot x^4 + \frac{b+c}{2} \cdot 1 + \left(\frac{b}{2} - \frac{c}{2}\right) \cdot (x+x^3).$$

KERNEL and Image: $T: V \to W$ linear, gives rise to 2 subspaces: Kernel of T, $\operatorname{Ker} T = \{v \text{ in } V \mid T(v) = 0\}$
$\operatorname{Im} T = \{T(v) \mid v \text{ in } V\}$.

Lemma: $\operatorname{Ker} T$, $\operatorname{Im} T$ are subspaces in V, W.

Pf: EASY;

EXP: Let A be an $m \times n$ mx; let $T: \mathbb{R}^n \to \mathbb{R}^m$; $T(x) = Ax$

Then:
$\operatorname{Ker} T = \{x \mid Ax = 0\} \stackrel{!}{=} \operatorname{null} A$;
$\operatorname{Im} T = \{T(x) \mid x \text{ in } \mathbb{R}^n\} \stackrel{!}{=} \{Ax \mid x \text{ in } \mathbb{R}^n\} = \operatorname{im} A$.

EXP: $T: M_{n,n} \to M_{n,n}$ $T(A) = A - A^T$. It is linear.

Find $\operatorname{Ker} T$, $\operatorname{im} T$

Sol: $\operatorname{Ker} T = \{A \mid A = A^T\} = \{A \mid A \text{ is symmetric}\}$

$\operatorname{Im} T = \{S \text{ in } M_{n,n} \mid T(A) = S \text{ for some } A \text{ in } M_{n,n}\} =$
$= \{S \mid S = A - A^T, \text{ for some } A \text{ in } M_{n,n}\} =$
$= \{S \mid S = -S^T\}$!!

Def: $T: V \to W$ is onto if $\operatorname{Im} T = W$

$T: V \to W$ is one-to-one if $T(v_1) = T(v_2) \Rightarrow v_1 = v_2$

EXP: $T: \mathbb{R}^2 \to \mathbb{R}$ $T\begin{pmatrix} x \\ y \end{pmatrix} = y$ is onto, not 1-1

$T: \mathbb{R} \to \mathbb{R}^2$ $T(x) = \begin{pmatrix} x \\ 0 \end{pmatrix}$ is 1-1, not onto

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THM: $T: V \to W$ linear. T is 1-1 $\iff$ $\operatorname{Ker} T = \{0\}$

Pf: EASY!!

DO IT by yourself ☺