TUE: March 10, 2009

(5.1) (last time: SUBSPACE TEST)

EXP: A $m \times n$ mx $\Rightarrow$
- Col A, Im A subspaces of $\mathbb{R}^m$
- null A subspace of $\mathbb{R}^n$; row A is a subspace
- a line, a plane through origin

EXP: V vector space $\Rightarrow$ $\{0\}$; V are subspaces. A proper subspace is a subspace different than 0, V.

EXP: Let v in V, then $\mathbb{R}v = \{r \cdot v \mid r \text{ \# in } \mathbb{R}\}$ is a subspace of V.

SOL: $0 = 0 \cdot v$ is in $\mathbb{R}v$; If x, y are in $\mathbb{R}v$, then $x = r_1 v$, $y = r_2 v$. So $x+y = r_1 v + r_2 v = (r_1 + r_2) v$ is in $\mathbb{R}v$; If x in $\mathbb{R}v$; c # $\Rightarrow$ $x = rv$ for some r, and hence we got $cx = c(rv) = (cr)v$ in $\mathbb{R}v$.

EXP: λ fixed #, $U = \{p(x) \text{ in } \mathbb{P} \mid p(\lambda) = 0\}$. Then U is a subspace of $\mathbb{P}$.

SOL: 0 is in U since $0(\lambda) = 0$; If $p(x), q(x)$ are in U, the $p(\lambda) = 0 = q(\lambda)$. Now note $(p+q)(\lambda) = p(\lambda) + q(\lambda) = 0 + 0 = 0$. So $p+q$ is in U; If c #, $p(x)$ in U, then $(cp)(\lambda) = c \cdot p(\lambda) = c \cdot 0 = 0$. So $cp(x)$ is in U.

VERY IMPT:

EXP: Let $v_1, v_2, \dots, v_n$ be vectors in V. If $a_1, a_2, \dots, a_n$ are #s, the element $a_1 v_1 + a_2 v_2 + \dots + a_n v_n$ IS CALLED a linear combination of the v_i. The set of all linear combinations of the v_i is called the **span** of the v_i, **denoted:** $\text{span}\{v_1, v_2, \dots, v_n\} =$

$= \{ a_1 v_1 + a_2 v_2 + \dots + a_n v_n \mid a_1, a_2, \dots, a_n \in \mathbb{R} \}$.

THM: Let $U = \text{span}\{v_1, v_2, \dots, v_n\}$ in V.

① U is a subspace of V containing $v_1, v_2, \dots, v_n$.

② U is the smallest subspace (of V) containing these elements (vectors).

Proof: As in $\mathbb{R}^n$.

EXP: Show $\mathbb{P}_n = \text{span}\{1, x, x^2, \dots, x^n\}$.

SOL: Let $p(x)$ be in $\mathbb{P}_n$; then: $p(x) = \overset{= a_0 \cdot 1}{a_0} + a_1 x + \dots + a_n x^n$ for some $a_0, a_1, \dots, a_n \in \mathbb{R}$. So $p(x)$ is in $\text{span}\{1, x, \dots, x^n\}$. **Done**.

EXP: $M_{2,2} = \left\{ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \middle| \; a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R} \right\} =$

$$= \left\{ A = a_{11} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{= E_{11}} + a_{12} \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{= E_{12}} + a_{21} \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{= E_{21}} + a_{22} \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{= E_{22}} \middle| \; a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R} \right\} = \text{span}\{E_{11}, E_{12}, E_{21}, E_{22}\}$$

DO: 8 a/300 ; 7 c/301 ; 9 a/301

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5.2 INDEPENDENCE and Dimension

capsin?!

Def: A set $\{v_1, v_2, \ldots, v_n\}$ in V (a vector space) is called **Linearly Independent** if: $r_1 v_1 + r_2 v_2 + \cdots + r_n v_n = 0$ implies that $r_1 = 0, r_2 = 0, \ldots, r_n = 0$.

DO: 2/309 a)

EXP: $\boxed{\text{If } v \neq 0 \text{ in } V, \text{ then } \{v\} \text{ is L.I.}}$ EASY proof...

THM: (FUNDAMENTAL Theorem) Let V be a vector space, suppose $V = \text{span}\{v_1, v_2, \ldots, v_n\}$; suppose that $\{u_1, u_2, \ldots, u_m\}$ is ~~any~~ a linearly independent set of V. Then $m \leq n$

Def A set of vectors $\{b_1, b_2, \ldots, b_n\}$ in V is called a basis if:
- $\{b_1, \ldots, b_n\}$ is LIN. IND
- $\text{span}\{b_1, b_2, \ldots, b_n\} = V$

Thm (Invariance Theorem) If $\{b_1, b_2, \ldots, b_n\}$ and $\{d_1, d_2, \ldots, d_m\}$ are 2 bases of V, then $n = m$.

So: **Def** If V is a non zero vector space, then the dimension of V, $\dim V$, is the # of element in a basis of V.

Convention: $\dim \{0\} = 0$ (0 is the zero space)

EXP: $\dim(\mathbb{P}_4) = 4+1$

SOL: BASIS: $1, x, \ldots, x^4 \Rightarrow$ DO IT!

EXP: $\dim M_{m,n} = m \cdot n$

SOL: The matrix units in $M_{m,n}$ form a basis.

$$E_{i,j} = \begin{array}{cc} & j^{th} \\ i^{th} & \begin{pmatrix} 0 & \vdots & 0 \\ \cdots & 1 & \cdots \\ 0 & \vdots & 0 \end{pmatrix} \end{array};$$

Def: A vector space V is called <u>finite dimensional</u> if it has a finite spanning set:

$$V = \text{span}\{v_1, v_2, \dots, v_n\}$$ for some elements (vectors) $v_1, v_2, \dots, v_n$.

EXP: $\mathbb{P}$ – the set of all polynomials is not finite dimensional.

Lemma (Independent Lemma) Let $\{v_1, v_2, \dots, v_n\}$ be a L.I. set of V; suppose v is not in $\text{span}\{v_1, v_2, \dots, v_n\}$. Then the larger set $\{v, v_1, v_2, \dots, v_n\}$ is L.I.

(Same proof as in $\mathbb{R}^n$).

Lemma (Dependent Lemma) A set of elements $\{v_1, v_2, \dots, v_n\}$ is L. Dependent $\Leftrightarrow$ one of the v_i is a linear combo of the others.

(Same proof!)

Of course: A set of vectors (elements) is L.D. if it is not L.I (By Definition!!!)

THM: Let $V \neq \{0\}$ be a finite dimensional vector space, and assume that V is spanned by n vectors.

1. V has a basis, and $\dim V \leq n$.
2. Every lin. Ind subset of V is a part of a basis
3. Every finite spanning set of V contains a basis.

THM: Suppose that $\dim V = \boxed{n}$. Let B be a set of $\boxed{n}$ vectors in V. Then:

B is L.I $\iff$ B spans V

In each case: B is a basis of V

THM: Let U be a subspace of a vector space V. Suppose $\dim V = n$.

1. U has a bases, and $\dim U \leq n$
2. If $\dim U = n$, then $U = V$
3. Every basis of U is a part of a basis of V.

DO:

3a/309 $U \subseteq V = P_2$. Since $\dim P_2 = 3 \Rightarrow \dim U \le 3$.

LI? $a(1-2x) + b(3x - x^2) + c(3-2x^2) = 0 \Rightarrow$

$$\begin{cases} a + 3c = 0 \\ -2a + 3b = 0 \\ -b - 2c = 0 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \xrightarrow{R_2 + 2R_1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$ So L.D.

In fact: $c = t$, $t\#$, $b = -2t$, $a = -3t$. Choose $t = 1$,

$3 - 2x^2 = 3(1-2x) + 2(3x - x^2) \Rightarrow U = \text{span}\{1-2x, 3x - x^2\}$ only!

L.I.? $a(1-2x) + b(3x - x^2) = 0$

$$\left[\begin{array}{cc|c} a & 0 & 0 \\ -2a & 3b & 0 \\ 0 & -b & 0 \end{array}\right] \Rightarrow a = 0, b = 0 \Rightarrow \text{L.I.}$$ So a basis is

$\{1-2x, 3x - x^2\} \Rightarrow \dim U = 2$

3c/309 $p(x) = p(-x) \Rightarrow a + bx + cx^2 = a - bx + cx^2$

$\Rightarrow b = 0 \Rightarrow U = \{p(x) \mid a + cx^2\} = \text{span}\{1, x^2\}$.

Since $\{1, x^2\}$ is (easily) LI $\Rightarrow \{1, x^2\}$ is a basis

$\Rightarrow \dim U = 2$

If time do: 3i/309 11/309 ; 17/309

$\boxed{21/310}$, 9/309, $\boxed{24\ a\ b}$/310

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