Lecture 16 — MARCH - 5 - 2009 - THU

(5.1) A vector space is a triple $(V, +, \cdot)$ such that: V is a nonempty set; $+, \cdot$ are 2 operations (addition, scalar multiplication) satisfy <u>10 Axioms</u>:

[(1) v, w in $V \Rightarrow v+w$ in V; (2) $v+w = w+v$; (3) $v+(w+u) = (v+w)+u$; (4) There is an element 0 in V such that $v+0=v$; (5) For each v in V, there is an element $-v$ in V s.t. $v+(-v)=0$; (6) if v is in V, c #, then cv <u>is</u> in V; (7) $c(v+w) = cv+cw$; (8) $(c+d)v = cv+dv$; (9) $c(dv) = (cd)v$; (10) $1 \cdot v = v$ (c, d #s)]

<u>EXP:</u> ($\mathbb{R}$, usual $+$, usual $\cdot$)

<u>EXP:</u> $M_{m,n}$ — the set of all $m \times n$ matrices, $+$ = addition of matrices; $\cdot$ = scalar multiplication; $0 = 0_{m \times n}$.

<u>EXP:</u> P — the set of all polynomials:

$P = \{ p(x) \mid p(x) = a_0 + a_1 x + \dots + a_n x^n,\ n \geq 0,\ a_0, \dots, a_n \text{ \#s} \}$.

0 = the zero polynomial; $+$ = usual addition of polynomials: $(3x + 7x^7) + (2x + 6x^2) = 5x + 6x^2 + 7x^7$;

$\cdot$ scalar multiplication: $c(3 + 7x + 8x^7) = 3c + 7cx + 8cx^7$.

<u>EXP:</u> $n \geq 0$; $P_n = \{ p(x) = a_0 + \dots + a_n x^n \mid a_0, \dots, a_n \text{ \#s} \}$ — the set of all polynomials of degree $\leq n$.

(SAME operations, when we add 2 polynomials of degree $\leq n$, we get a polyn. of deg $\leq n$.

EXP: Let D be a set; Define $\mathbb{F}(D) = \{ f : D \to \mathbb{R} \mid f$ functio

$+$: f, g in $D \rightsquigarrow f+g : D \to \mathbb{R}$; $(f+g)(x) =$

$f(x) + g(x)$; x in D.

$\cdot$: $c\#$, f in $D \rightsquigarrow cf : D \to \mathbb{R}$; $(cf)(x) = cf(x)$ x in D

The zero element is $f_0 : D \to \mathbb{R}$, $f_0(x) = 0$, x in D

EXP: $\{0\}$; Define $0 + 0 = 0$; $c0 = 0$

PROPERTIES: (1) Cancellation: $w + v = u + v \Rightarrow w = u$

(2) $0 \cdot V = 0$; for any V in V; (3) $c \cdot 0 = 0$; (4) $aV = 0 \Rightarrow$ either $a = 0$ or $V = 0$; (5) $(-1)V = -V$; (6) $(-cV) = -(cV) =$ $= c(-V)$.

DO:

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1 a, b, (c)! /300

2 a NOT: $2[x \; y \; z] = [2x \; 2y \; 2z]$

$\qquad\quad \parallel$

$(1+1)[x \; y \; z] = [x \; y \; z] + [x \; y \; z] = [2x \; 2y \; z]$

$\Rightarrow 2z = z$ ↯

3 e/300 $V = \{ [x \; y] \mid x^2 = y^2 \}$; $(+, \cdot) \to M_{1,2}$

NOT: $[1, -1]$ in V; $[1 \; 1]$ in $V \Rightarrow$ sum not in V:

$[2 \; 0]$ not in V since $4 \neq 0$.

$\boxed{3\,h/300}$ $2f = f + f \Rightarrow (2f)(x) = (f+f)(x) \Rightarrow$

$f(2x) = f(x) + f(x) \Rightarrow f(2x) = 2f(x)$ Choose $f(x) =$

$= 2x + 1$ or $f(x) = 1$, $\Rightarrow$ ↯

§ SUBSPACES

Def: if V is a vector space, a nonempty subset $U \subseteq V$ is called a subspace of V if U itself is a vector space using $+, \cdot$ of V.

EXP: $\mathbb{P}_n$ is a subspace of $\mathbb{P}$; $\mathbb{P}$ is a subspace of $\mathbb{F}(\mathbb{I})$

THM (SUBSPACE TEST) A subset U of a vector space V is a subspace $\Longleftrightarrow$ ① 0 is in U; ② u, v in U $\Rightarrow$ $u+v$ in U; ③ c #, u in U $\Rightarrow$ cu in U.

EXP: A $m \times n$ $\Rightarrow$
- $\cdot$ Col A, Im A subspaces of $\mathbb{R}^m$
- $\cdot$ null A subspace of $\mathbb{R}^n$; row A is a subspace ...
- $\cdot$ a line, plane through Origin

EXP: V vectorspace $\Rightarrow$ $\{0\}$, V are subspaces

A proper subspace is a subspace different than 0, V.

EXP: Let v in V, then $\mathbb{R}v = \{ rv \mid r \text{ in } \mathbb{R} \}$ is a subspace of V.

SOL: $0 = 0 \cdot v$ is in $\mathbb{R}v$; if x, y are in $\mathbb{R}v$, then $x = r_1 v$; $y = r_2 v$. So $x + y = r_1 v + r_2 v = (r_1 + r_2) v$ is in $\mathbb{R}v$; If c #, x in $\mathbb{R}v$; then $x = r \cdot v$; r #. So $cx = c(rv) = (cr)v$ is in $\mathbb{R}v$.

EXP: λ fixed #; let $U = \{ p(x) \text{ in } \mathbb{P} \mid p(\lambda) = 0 \}$.

Then U is a subspace of $\mathbb{P}$.

SOL: 0 is in U since $0(\lambda) = 0$.

If $p(x), q(x)$ are in U, then $p(\lambda)=0=q(\lambda)$. Now $(p+q)(x) = p(x)+q(x) \Rightarrow (p+q)(\lambda) = p(\lambda)+q(\lambda) = 0+0 = 0$. So $p+q$ is in U, If c #, $p(x)$ in U, then $(cp)(x) = c \cdot p(x)$ So $(cp)(\lambda) = c \cdot p(\lambda) = c \cdot 0 = 0$. So $cp(x)$ is in U.

VERY IMPORTANT:

EXP: Let $v_1, \dots, v_n$ be vectors in V. If $a_1, \dots, a_n$ are #s, the element $a_1 v_1 + \dots + a_n v_n$ is called a linear combination of the v_i. The set of all the linear combinations of the v_i is called the SPAN of the v_i, denoted: $\text{span}\{v_1, \dots, v_n\} = \{a_1 v_1 + \dots + a_n v_n \mid a_1, \dots, a_n \text{ \#s}\}$.

$\boxed{\text{THM}}$ Let $U = \text{span}\{v_1, \dots, v_n\}$ in V.

(1) U is a subspace of V containing $v_1, v_2, \dots, v_n$

(2) U is the smallest subspace containing these vectors.

Pf: (As in $\mathbb{R}^n$)

EXP: Show $P_n = \text{span}\{1, x, x^2, \dots, x^n\}$

SOL: Let $p(x)$ be in P_n, then $p(x) = \underset{a_0 \cdot 1}{a_0} + a_1 x + \dots + a_n x^n$ is in $\text{span}\{1, x, \dots, x^n\}$.

EXP: $M_{2,2} = \left\{ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \middle| \; a_{11}, \ldots, a_{22} \;\#s \right\}$

$= \left\{ A = a_{11} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_{11}} + a_{12} \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E_{12}} + a_{21} \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{E_{21}} + a_{22} \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_{22}} \middle| \right.$

$\left. a_{11}, \ldots, a_{22} \;\#s \right\} = \text{span}\{E_{11}, E_{12}, E_{21}, E_{22}\}$

8/a/300 $\quad f, g$ in $U \Rightarrow f(0)=1, \quad g(0)=1$

$(f+g)(0) = f(0)+g(0) = 1+1 = 2 \neq 1 \Rightarrow f+g$ NOT in U

So: NOT.

8 e/300, $\quad$ ✓ 0 is in U; $\quad 0(0)=0 = 0(1)$

✓ f, g in $U \Rightarrow \begin{cases} f(0)=f(1) \\ g(0)=g(1) \end{cases}$; then $(f+g)(0) = f(0)+g(0)$

$= f(1)+g(1) = (f+g)(1)$. ✓

f in U, $c\#$. So $f(0)=f(1)$. Now $(cf)(0) = cf(0) =$

$= cf(1) = (cf)(1)$. ✓

7c/301 $\quad U = \{A \mid A = -A^T\} \subseteq M_{2,2}$.

0 is in U; $-(A+B)^T = -(A^T+B^T) = -A^T - B^T = A+B$

$-(cA)^T = -(cA^T) = -cA^T = -(-cA)$ ✓ $\quad$ <u>YES</u>

7e/301 $\quad U = \{A \mid \det A = 0\} \subseteq M_{2,2}$

<u>SOL</u>: $U \ni \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; $U \ni \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow U \not\ni \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad$ <u>NOT!</u>

9 a/301 Is $V_1 = x^2+1$, or $V_2 = x^2-2x-6$ in span$\{u, w\}$; where $u = x^2-2$; $w = x^2+x$.

SOL: ⊛ Can we solve $au + bw = V_1$?

$a(x^2-2) + b(x^2+x) = x^2+1 \Rightarrow x^2(a+b) + bx - 2a$

$= x^2 + 0\cdot x + 1 \Rightarrow \begin{cases} a+b=1 \\ b=0 \\ -2a=1 \end{cases} \Rightarrow \begin{cases} a=1 \\ b=0 \\ a=-\frac{1}{2} \end{cases} \Rightarrow$ No Sol

so: No SOL $\Rightarrow$ V_1 NOT in the span.

⊛ Can we solve $au + bw = V_2$?

$a(x^2-2) + b(x^2+x) = x^2-2x-6 \Rightarrow \begin{cases} a+b=1 \\ -2=b \\ -2a=-6 \end{cases}$

$\Rightarrow \begin{cases} b=-2 \\ a=3 \end{cases}$ $\boxed{\text{YES}}$ $\boxed{3\cdot u + (-2)w = V_2}$

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24 a/301 Can I find a, b, c, d such that

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}?$$

SOL: $\begin{cases} a_{11} = a+b \\ a_{12} = c+d \\ a_{21} = c \\ a_{22} = b \end{cases} \Rightarrow \begin{cases} a = a_{11} - a_{22} \\ b = a_{22} \\ c = a_{21} \\ d = a_{12} - a_{21} \end{cases}$ Since

we got SOL $\Rightarrow$ the answer is YES.