Lec. 13 : TUE

$\boxed{\S 4.2}$ DO 46/194   55/194 Use Thm 2/192 ⟵ LAST lectu

Def: A set $\{X_1, \dots, X_k\}$ is called LINEARLY Dependent if it is not Lin. Independent.

THM 3 A set $\{X_1, \dots, X_k\}$ is L.D if and only if one of the vectors X_i is a linear combo of the others.

Pf: $\Rightarrow$ W.L.O.G we may assume: $t_1 X_1 + \dots + t_k X_k = 0$, $t_1 \neq 0$.

So: $X_1 = -\frac{t_2}{t_1} X_2 - \dots - \frac{t_k}{t_1} X_k$.

$\Leftarrow$ EASY !!

COR: Let $\vec{v}, \vec{w}$ be 2 non-zero vectors (in $\mathbb{R}^3$);

① $\{\vec{v}, \vec{w}\}$ are L.D $\iff$ $\vec{v} \parallel \vec{w}$.

② $\{\vec{v}, \vec{w}\}$ are L.I. $\iff$ $\vec{v}$ is NOT parallel to $\vec{w}$.

SO: span$\{\vec{v}, \vec{w}\}$ is either a line or a plane (passing through the origin).

DO : 9 a/194 , 7 a/194

§ 4.3 DIMENSION

THM 1 (FUNDAMENTAL THM) Let U be a subspace of $\mathbb{R}^n$. If U is spanned by m vectors, and if U contains k linearly independent vectors, then $k \le m$.

COR • No linearly Independent set in $\mathbb{R}^n$ can contain more than n vectors. (WHY? A: The columns of I_n span $\mathbb{R}^n$!!!)

- No spanning set of $\mathbb{R}^n$ can contain fewer than n vectors (Use again Thm 2)

(We are lead to:)

DEF: A set $\{X_1, \dots, X_k\}$ of vectors in U is called a basis if:
① $\{X_1, \dots, X_k\}$ is L.I. ; ② span $\{X_1, \dots, X_k\} = U$

By Thm 1

THM 2 (Invariance Theorem) if $\{X_1, \dots, X_k\}$ and $\{Y_1, \dots, Y_m\}$ are 2 BASES of U, then $k = m$.

(EASY!!)

Def: The number of vectors in $\boxed{\text{a basis}}$ of U is called the dimension of U. Denoted: $\dim U$.

$\boxed{\text{EXP 1}}$ $\dim \mathbb{R}^n = n$

Pf: The Columns of I_n form a basis

(Use Thm 2/192 ...)

Def: $\dim\{0\} = 0$

EXP: LINE in $\mathbb{R}^3$: $L = \{t\vec{d} \mid t \ \#\}$, where $\vec{d} \neq \vec{0}$ is a direction vector. BASIS $\{\vec{d}\}$.

So $\dim L = 1$

EXP: if $\{X_1, X_2, \dots, X_k\}$ is a basis for U, and if $a_1, \dots, a_k$ are non-zero #s, then $\{a_1 X_1, \dots, a_k X_k\}$ is a basis of U.

SOL: ← just do it @ blackboard: easy....

DO: 2/201 a; b $\quad \frac{3a \ / \ \text{NO}}{201}$ $\quad$ @ $\boxed{3 \neq 4}$

§4.4 RANK

Def Let A be an $m \times n$ mx: < FIND BASIS for $\{$null A, im A $\}$

- The column space, col A, is the subspace of $\mathbb{R}^m$ spanned by the columns of A.
- The row space, row A, of an $m \times n$ mx. A, is the subspace of $\mathbb{R}^n$ spanned by the Rows of A

(We regard the Rows of A as elements of $\mathbb{R}^n$)

Lemma 1 1) If $A \mapsto B$ using a sequence of Row Op., then
row B = row A

2) if $A \mapsto B$ using a sequence of Column Op, then
col B = col A

THM rank (A) = dim (row A)
if $A \to R$ (a Row Echelon Form mx.), then the non-zero Rows of R are a basis of Row A

Pf: By above $(B=R)$ and by an exc. last lec. ⇒ the second statement. So now we get the first one !!

Do 4 e/ 210

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & -3 & 1 & 4 \\ 4 & -13 & 5 & 6 \end{pmatrix} \xrightarrow[R_3-4R_1]{R_2-2R_1} \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 3 & -2 \\ 0 & -21 & 9 & -6 \end{pmatrix} \xrightarrow{R_3-3R_2}$$

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So: $\{(1 \; 2 \; -1 \; 3)^T; (0,-7,3,-2)^T\}$ is a basis!

THM (RANK Theorem) Let A be an $m \times n$ mx. Then:

$$\dim(\mathrm{Row}\, A) = \dim(\mathrm{Col}\, A) = \mathrm{rank}\, A$$

PLAN to get a basis for col A:

$A \longrightarrow \longrightarrow R$ (r.e.f. mx); circle the ~~corresponding~~ Columns that contain the leading 1's, go back to A and select the corresponding columns: They form a basis for col A.

DO: Find the following: A basis for col A,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 20 \end{pmatrix}$$

COR 1 If A is $m \times n$, then $\mathrm{rank}\, A \le m$, $\mathrm{rank}\, A \le n$

COR 2 An $n \times n$ mx. is invertible $\iff$ $\mathrm{rank}\, A = n$

COR 3 $\mathrm{rank}(A^T) = \mathrm{rank}\, A$ for any mx. A. $\boxed{\mathrm{Col}(A^T) = \mathrm{row}\, A \text{ etc.}}$

$\boxed{\text{COR 4}}$ $\mathrm{rank}\, A = \mathrm{rank}\,(UAV)$ for any invertible matrices U and V.

$\boxed{\text{THM 3}}$ Let A be an $m \times n$ mx. TFAE:

1. $Ax = 0$ has only the trivial solution ($x = 0$)
2. The columns of A are L.I.
3. $\mathrm{rank}\, A = n$
4. $A^T A$ = invertible

$\boxed{\text{DO: } 1 \to 2 \to 3,\ 4 \to 1}$ (and its sister :)

THM4/207 Let A be an $m \times n$ mx. TFAE:

(1) $AX = B$ has a solution for every B in $\mathbb{R}^m$.

(2) The columns of A span $\mathbb{R}^m$.

(3) rank $A = m$

(4) $A \cdot A^T$ is invertible

DO: 10/210 NO

$A = m \times n$

By Thm 3 $\Rightarrow$ rank $= n$
Rows = Ind $\Rightarrow$ dim Row $A = n$
$\Rightarrow m = n$ ↯

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§5 Null space and Image

Thm (5) Let A be an $m \times n$ mx. Then:

$$\text{col}\,A = \text{Im}\,A = \{ AX \mid X \text{ in } \mathbb{R}^n \}$$

so: $\dim(\text{Im}\,A) = \dim(\text{Col}\,A) = \text{rank}\,A$.

So you can get from now on basis for $\boxed{\text{Im}\,A}$

what about null A??

(THM) Let A be $m \times n$; rank $A = r$; let $X_1, \ldots, X_{n-r}$ be the basic solutions (from $AX = 0$).

Then $\{ X_1, \ldots, X_{n-r} \}$ is a basis of null A;

hence: $\dim(\text{null}\,A) = n - r$

$\boxed{\text{So}}$ COR: $\dim(\text{Im}\,A) + \dim(\text{Null}\,A) = n$

DO 11/210

$\dim \text{Im}\,A + 2 = 6$

$3 \geq \dim \text{Im}\,A = 4$

↯

#6/ if time!!!