Lec: 12 : -THU-

§4.1. Recall: span $\{X_1, \ldots, X_k\} = \{t_1 X_1 + \cdots + t_k X_k \mid t_1, \ldots, t_k$ are #s$\}$.

4c/187 (find null A)

Then DO: 10, 13 /188

DEF: if a subspace U has the form: $U = \text{span}\{X_1, \ldots, X_k\}$ we say that $X_1, \ldots, X_k$ are a spanning set of U {or: U is spanned by the X_is}.

EXP: $\mathbb{R}^n = \text{span}\{E_1, E_2, \ldots, E_n\}$, where $E_1, E_2, \ldots, E_n$ are the columns of I_n.

Pf....

EXP: (!) Let A be an $m \times n$ mx. Let $\{X_1, X_2, \ldots, X_k\}$ be the basic solutions of $AX=0$. Then:

$$\text{null } A = \text{span}\{X_1, \ldots, X_k\}$$

(Recall: §1.3)

EXP(!!) if $A = [C_1 \ C_2 \ \cdots \ C_n]$, then $\text{im } A = \text{span}\{C_1, C_2, \ldots, C_n\}$.

SOL: $Y = AX = [C_1 \ C_2 \cdots C_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 C_1 + \cdots + x_n C_n$ in $\text{span}\{C_1, \ldots, C_n\}$.

DO: 15a/189; 18 → easy; $U = \text{null}(A-B)$ →

14/189; 8 d,c/188; 24/a/188

$\boxed{\text{4c/188 Im } A}$

§4.2 LINEAR INDEPENDENCE

Def A set $\{X_1, \dots, X_k\}$ of vectors is called LINEARLY INDEPENDENT if $t_1 X_1 + t_2 X_2 + \dots + t_k X_k = 0 \Rightarrow$

$t_1 = 0,\ t_2 = 0, \dots, t_k = 0$.

THM 1 if $\{X_1, X_2, \dots, X_k\}$ is L.i., then every vector X in $\operatorname{span}\{X_1, X_2, \dots, X_k\}$ has a unique representation as a linear combo of the X_i's.

Pf: if $t_1 X_1 + \dots + t_k X_k = r_1 X_1 + \dots + r_k X_k \Rightarrow$
$(t_1 - r_1) X_1 + \dots + (t_k - r_k) X_k = 0 \xRightarrow{L.i.} t_1 - r_1 = 0, \dots,$
$t_k - r_k = 0 \Rightarrow \boxed{t_1 = r_1, \dots, t_k = r_k}$.

DO: 1 a,d /193 (L.i.)

EXP: The columns of I_n are L.i. (in $\mathbb{R}^n$).

EXP: Any set of vectors that contains 0 is NOT L.i.

EXP The NON-ZERO ROWS of any REF mx are L.i.

THM 2 The following are equivalent: (for $n \times n$ mx. A)

1. A is invertible
2. The columns of A are L.i (in $\mathbb{R}^n$)
3. The columns of A span $\mathbb{R}^n$.
4. The rows of A are L.i (in $\mathbb{R}^n$)
5. The rows of A span $\mathbb{R}^n$

EASY Pf....

EXC: 4b/194 ; 5/194 [Use the previous Thm) : 2]

DEF: A set $\{x_1, --, x_k\}$ is called <u>Linearly Dependent</u> if it is NOT L. Ind.

THM3) A set $\{x_1, --, x_k\}$ is <u>L. D.</u> if and only if one of the vectors x_i is a <u>linear</u> combo of the others.

<u>Pf:</u> $\Rightarrow$ WLOG: $t_1 x_1 + \cdots + t_k x_k = 0$, $t_1 \neq 0$. So

$$x_1 = -\frac{t_2}{t_1} x_2 - \cdots - \frac{t_k}{t_1} x_k.$$

$\Leftarrow$ <u>EASY</u>;

<u>COR:</u> Let $\vec{v}, \vec{w}$ be nonzero vectors in $\mathbb{R}^3$

(1) $\{\vec{v}, \vec{w}\}$ are L. D $\Leftrightarrow$ $\vec{v} \parallel \vec{w}$

(2) $\{\vec{v}, \vec{w}\}$ are L.I. $\Leftrightarrow$ $\vec{v}$ NOT parallel to $\vec{w}$.

<u>So:</u> span$\{\vec{v}, \vec{w}\}$ is either a line or a plane (passing through the ORIGIN).

9a/194 ; 7a/194;

(4.3) <u>DIMENSION</u>

THM1 (<u>FUNDAMENTAL THM</u>) Let U be a subspace of $\mathbb{R}^n$. IF U is spanned by m vectors, <u>and</u> U contains k linearly independent vectors, THEN $k \leq m$.

COR: • No LIN. IND set in $\mathbb{R}^n$ can contain more than n vectors (WHY? A: the columns of I_n span $\mathbb{R}^n$)

• No spanning set of $\mathbb{R}^n$ can contain fewer than n vectors (SAME answer !!)

DEF: A set $\{x_1, \dots, x_k\}$ of vectors in U is called a BASIS of U if:

1. $\{x_1, \dots, x_k\}$ is L.I.
2. span $\{x_1, \dots, x_k\} = U$.

SO: THM2 (Invariance Thm) if $\{x_1, x_2, \dots, x_k\}$ and $\{y_1, \dots, y_m\}$ are 2 bases of U, then $k = m$.

Thm1

(EASY!)

DEF The number of vectors in $\boxed{\text{a BASIS}}$ of U is called the dimension of U. Denoted dim U.

EXP1 $\dim \mathbb{R}^n = n$.

Pf: the columns of I_n form a basis.

Def $\dim \{0\} = 0$.

EXP: Line in $\mathbb{R}^3$: $L = \{t\vec{d} \mid t \#\}$, $\vec{d}$ dir. vect, $\neq \vec{0}$.

BASIS $\{\vec{d}\} \Rightarrow \dim L = 1$.

EXP: if $\{x_1, x_2, \dots, x_k\}$ is a basis for U, and if $a_1, \dots, a_k$ are nonzero #s, then: $\{a_1 x_1, \dots, a_k x_k\}$ is a BASIS of U

DO: 2/201 a, b   3a/201 NO: 3 + 4