- INTRODUCE YOURSELF
- EXPLAIN / DISTRIBUTE $\longrightarrow$ OUTLINE

§2.5 COMPLEX NUMBERS

Can we solve $x^2+1=0$? IN real #s ($\mathbb{R}$): NO!

Def Let i be a # (non real) s.t. $i^2=-1$

A complex # is a # of the form: $z=\boxed{a+bi}$,

where a, b are real #s.

$\boxed{\text{EXP:}}$ $3-i$; $4+\frac{2}{3}i$, $\sqrt{2}+\sqrt{3}\,i$

NOTATION $\mathbb{C}$

Def Given $z=a+bi$, a is called The Real Part of z, b is called the Imaginary part of z.

NOTE: Every real # is a complex #:

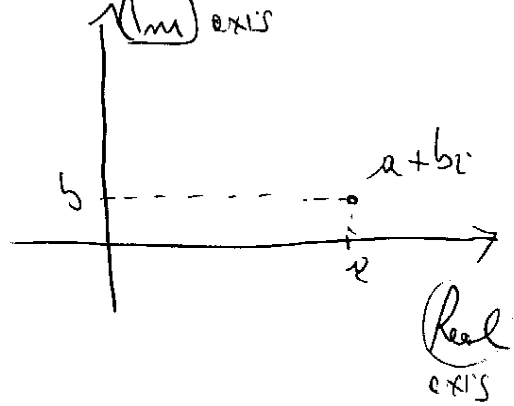
a in $\mathbb{R}$ $\longrightarrow$ $a=a+0.i$

Def: 2 complex #s are equal if and only if they have the same real and imaginary parts:

$2+3i \neq -2+3i$

GEOMETRICAL DISPLAY (of complex #s)

We identify the complex # $z=a+bi$ with the point $P(a,b)$:



$\rightarrow$ DRAW $\boxed{-1-i}$

Def The plane is called the COMPLEX PLANE

OPERATIONS: $z = a+bi$; $w = c+di$

Addition (+) $\quad z + w = (a+c) + (b+d)i$

Multiplication ($\cdot$) $\quad zw = (ac-bd) + (ad+bc)i \quad (\text{Use } i^2=-1)$

EXP: $(3-7i) + (-1+9i) = 2 + 2i$

$(3-7i)(-1+9i) = (-3+63) + (27+7)i =$

$= 60 + 34i.$

Def: if $z = a+bi$ is a Complex #, the conjugate of z, denoted $\bar{z}$, is given by $\bar{z} = a-bi$.

EXP: $\overline{-3-2i} = -3+2i$; $\bar{4} = 4$

PROPERTIES:

$$\begin{cases} \overline{z \pm w} = \bar{z} \pm \bar{w} \\ \overline{zw} = \bar{z}\,\bar{w} \\ \bar{\bar{z}} = z \\ z \text{ is real} \iff z = \bar{z} \end{cases} \qquad (\Leftarrow \text{TRY IT})$$

Def: if $z = a+bi$ is a complex # (in $\mathbb{C}$) then the absolute value of z, denoted $|z|$, is the positive or zero real #: $\sqrt{a^2+b^2}$.

NOTE: $|z|^2 = z\cdot\bar{z} = a^2+b^2$

PROPERTIES:

$$\begin{cases} |z| \geq 0 \\ |z| = 0 \text{ iff } z = 0 \\ |zw| = |z|\cdot|w| \end{cases}$$

Division ($\div$) if $z \neq 0$, then $\frac{1}{z} = z^{-1} = \frac{\bar{z}}{|z|^2}$

EXP: Find $\frac{4-3i}{3+5i}$

SOL:
$$\frac{4-3i}{3+5i} = \frac{4-3i}{3+5i} \cdot \frac{3-5i}{3-5i} = \frac{(12-15)+(-20-9)i}{34}$$

$$= -3/34 + (-29/34)i.$$

QUADRATICS / ROOTS SOLVE $ax^2+bx+c=0$; $a \neq 0$

SOL: $x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

1°) $b^2-4ac \rightarrow$ 2 distinct roots

2°) $b^2-4ac=0 \rightarrow$ one (repeated) root

3°) $b^2-4ac<0 \rightarrow$ 2 conjugated complex #s as Roots

EXP: SOLVE: $x^2+x+100=0$

SOL: $x_{1,2} = \frac{-1 \pm \sqrt{1-400}}{2} \Rightarrow \begin{cases} x_1 = \frac{-1-\sqrt{-399}}{2} \\ x_2 = \frac{-1+\sqrt{-399}}{2} \end{cases}$

$$\Rightarrow \begin{cases} x_1 = -1/2 - \frac{i\sqrt{399}}{2} \\ x_2 = -1/2 + \frac{i\sqrt{399}}{2} \end{cases}$$

IN CASE 3°) the quadratic is called irreducible.

EXP: If $z = 7-i$, find an irreducible quadratic with z as a root.

SOL: $(x-(7-i))(x-(7+i)) = \dots$ since $(7+i)$ is also a root!!

FUNDAMENTAL Theorem of ALGEBRA

(T) Every complex polynomial $f(x)$ of degree $n \geq 1$ has the form: $f(x) = u \cdot (x-\lambda_1)(x-\lambda_2) \cdots (x-\lambda_n)$ for some complex #s: $\lambda_1, \lambda_2, \ldots, \lambda_n \to$ the roots of $f(x)$; $u \neq 0$ is the coeff of x^n in $f(x)$.

EXC: 5 a/119

3 c)/119 $(a+bi)(1+i) = a-bi - 3-2i \Rightarrow z = \ldots$

7/119 (T/F) a) T ; b) F $z=i$, $w=1$ c) (T) ; d) $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

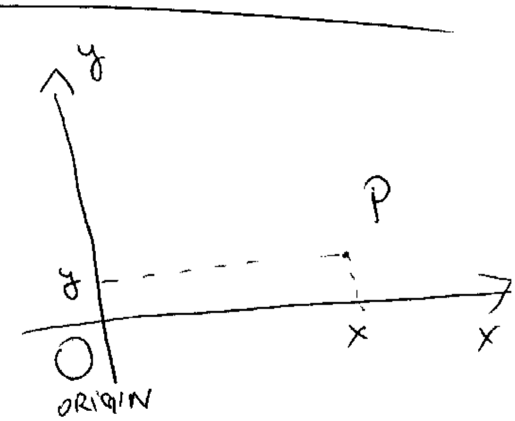
e) $|1+0\cdot i| = |0+1\cdot i|$, but $1 \neq i$: (F)

f) $a-bi = -a-bi \Rightarrow a=0$ (T)

CH:3 VECTOR GEOMETRY

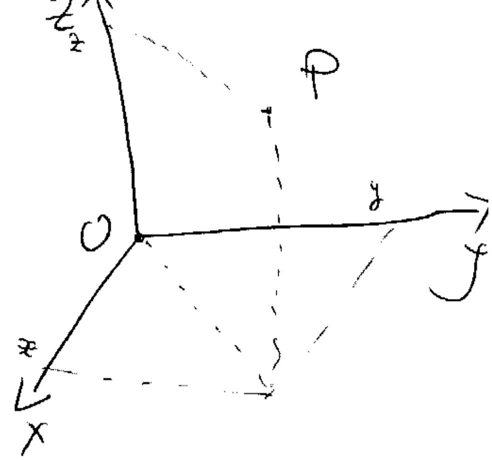
§3.1 Geometric vectors

a) The plane $\begin{cases} \text{Origin} \\ \text{2 perpendicular axes} \\ \text{scale} \end{cases}$



Each point P determines a unique pair (x,y) of #s, called the coordinates; we write $P(x,y)$

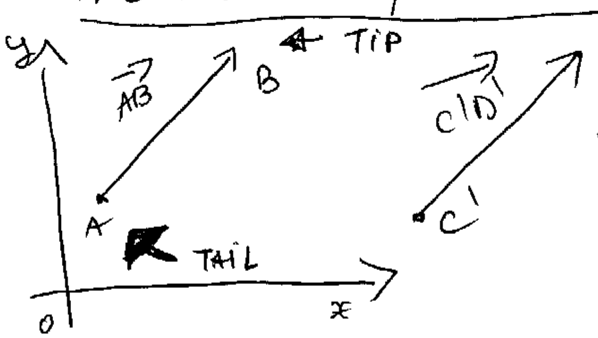
b) The Space $\begin{cases} \text{- origin} \\ \text{- 3 mutually } \perp \text{ axes} \\ \text{- scale} \end{cases}$

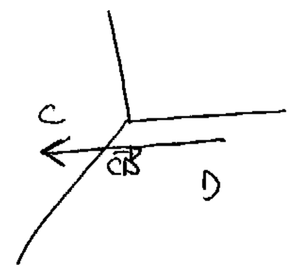


Each point P determines a unique triple (x, y, z) of #s, the coordinates of P; we write $P(x, y, z)$.

$\boxed{\text{VECTORS}}$ ≡ arising from Geometry

Def if A, B are 2 points (in plane/space), the directed segment from A to B is called the vector from A to B, denoted $\overrightarrow{AB}$.



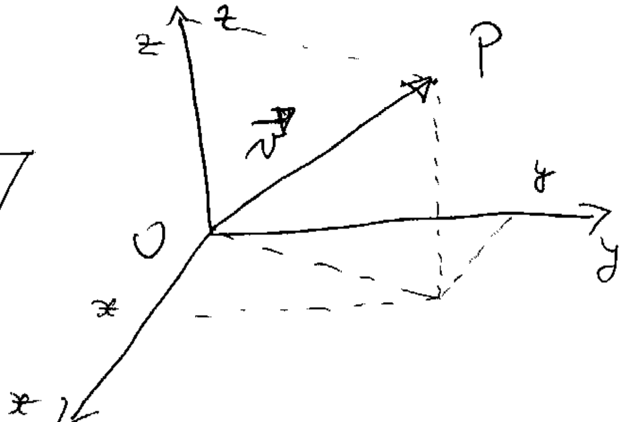


$\boxed{\text{Def: 2 vectors are equal if they have the same length and the same direction}}$

EXP: so $\overrightarrow{AB} = \overrightarrow{C'D'}$

Def The length of $\overrightarrow{AB}$ is denoted $\|\overrightarrow{AB}\|$.

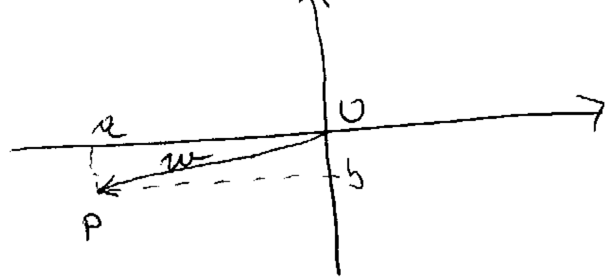
Def A vector $\vec{v}$ is in STANDARD POSITION if its $\boxed{\text{TAIL}}$ is in the origin.



P is determined by x, y, z; we may write: $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \overrightarrow{OP}$

- it's called matrix form -

In plane : $\vec{OP} = \vec{w} = \begin{bmatrix} a \\ b \end{bmatrix}$.



OPERATIONS:

+ Addition $\quad \vec{v} + \vec{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x+x' \\ y+y' \\ z+z' \end{pmatrix}$

- (scalar multiplication) $\quad k\vec{v} = k\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} kx \\ ky \\ kz \end{pmatrix}$
- The zero vector : $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
- The negative $\quad -\vec{v} = -\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$

PROPERTIES:

$$\begin{cases} \vec{u} + \vec{v} = \vec{v} + \vec{u} \\ (\vec{u}+\vec{v}) + \vec{w} = \vec{u} + (\vec{v}+\vec{w}) \\ \vec{0} + \vec{v} = \vec{v} \\ \vec{v} + (-\vec{v}) = \vec{0} \end{cases}$$

$a, b = \#s$

$$(a+b)\vec{v} = a\vec{v} + b\vec{v}$$
$$a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$$
$$(ab)\vec{u} = a(b\vec{u})$$
$$1 \cdot \vec{v} = \vec{v}$$

(T) Let $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$; $\vec{w} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ be vectors.

(1) $\vec{v} = \vec{w}$ iff $\quad x = x_1, \; y = y_1, \; z = z_1$

(2) $\|\vec{v}\| = \sqrt{x^2+y^2+z^2}$

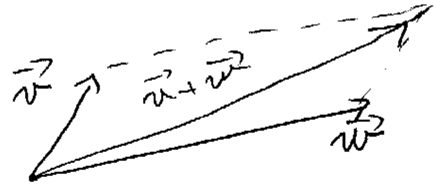
(3) $\vec{v} = \vec{0} \iff \|\vec{v}\| = 0$

(4) $\|a\vec{v}\| = |a| \cdot \|\vec{v}\|$; a scalar (#)

NOTE: If $\vec{v}$ is a vector, $-\vec{v}$ has the same length but opposite direction.

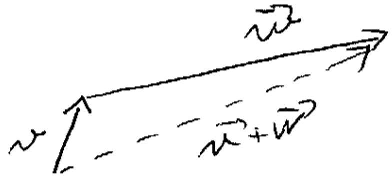
Vector Addition

- Parallelogram LAW

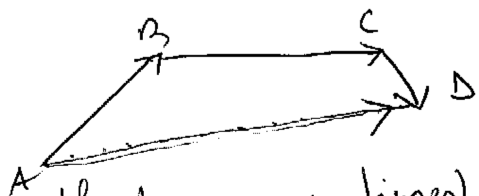


OR

- TIP-TO-TAIL Method:



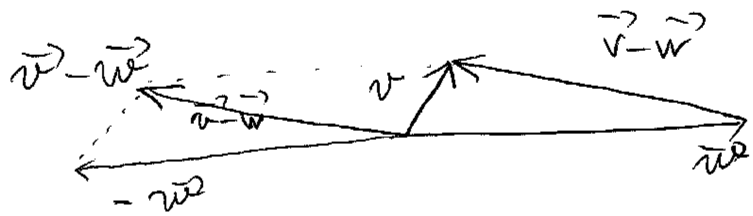
IMPORTANT NOTE:



$\vec{AB}+\vec{BC}+\vec{CD}=\vec{AD}$

(use the law/method many times)

Difference:



THM Let $P_1(x_1,y_1,z_1)$; $P_2(x_2,y_2,z_2)$ be 2 points. Then:

(1) $\overrightarrow{P_1P_2}=\begin{bmatrix} x_2-x_1 \\ y_2-y_1 \\ z_2-z_1 \end{bmatrix}$; (2) $d(P_1,P_2)=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

(Py T's thm.)

(In plane: (1)' $\overrightarrow{P_1P_2}=\begin{bmatrix} x_2-x_1 \\ y_2-y_1 \end{bmatrix}$; $d(P_1,P_2)=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

where $P_1(x_1,y_1)$; $P_2(x_2,y_2)$

Def: Two non-zero vectors are parallel if they have the same or opposite direction

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THM Let $\vec{v},\vec{w}$ be 2 nonzero vectors.

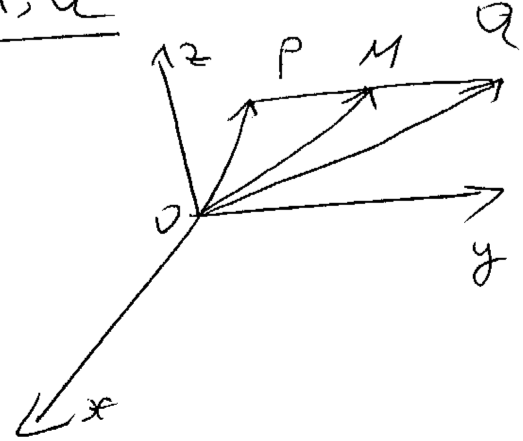
TFSAE: (1) $\vec{v},\vec{w}$ are parallel; (2) one of $\vec{v},\vec{w}$ is a sc. multiple of the other.

DO: 11/141

§ MID POINT between 2 points P, Q

$\vec{OM} = \frac{1}{2}(\vec{OP} + \vec{OQ})$

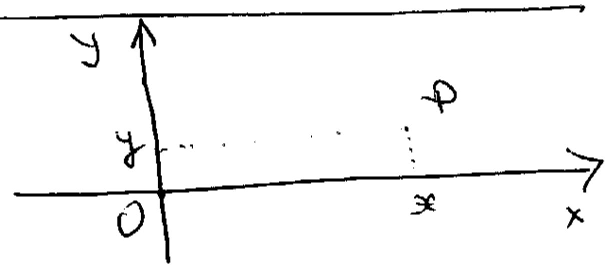
(half of the diagonal)



DO: 10/141 c) (NOT)

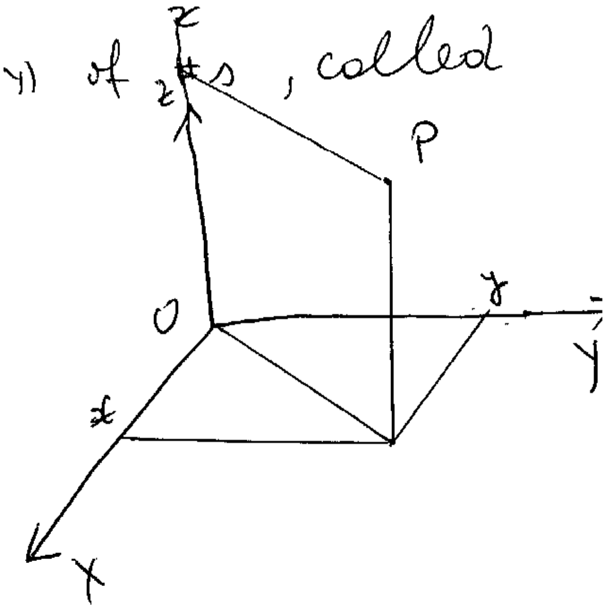
CH3 VECTOR GEOMETRY

§3.1 Geometric Vectors



a) THE PLANE { ORIGIN; 2 perpendicular axes; scale

Each point P determines a pair (x,y) of #s, called the coordinates; we write $P(x,y)$.

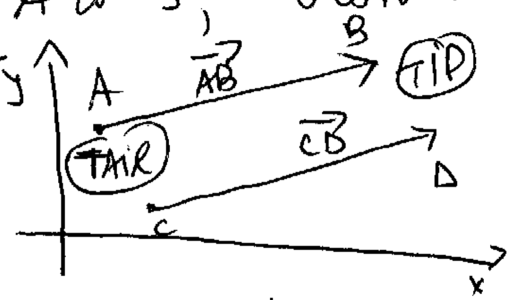


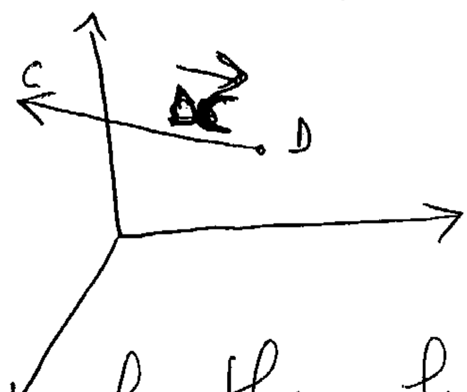
b) THE SPACE { ORIGIN; 3 mutually $\perp$ axes; scale

Each point P determines a triple (x,y,z) of #s, the coordinates of P. We write $P(x,y,z)$

Vectors (arising from geometry)

Def if A, B are 2 points (plane/space), the directed segment from A to B is called the vector from A to B, denoted $\overrightarrow{AB}$.





Def(=) 2 vectors are equal if they have the same length and direction.

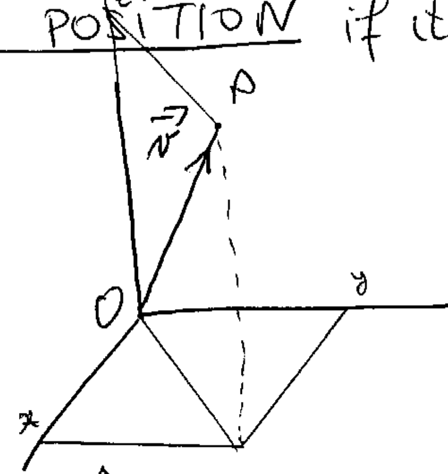
EXP: $\overrightarrow{AB} = \overrightarrow{CD}$

Def $(\|\cdot\|)$ The length of $\overrightarrow{AB}$ is denoted by $\|\overrightarrow{AB}\|$.

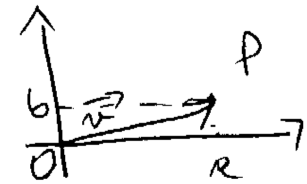
Def: A vector $\vec{v}$ is in <u>STANDARD POSITION</u> if its TAIL is in the origin:

$\vec{v} = \vec{OP}$.

P is determined by x, y, z; we may write $\vec{v} = \vec{OP} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ (mx. <u>notation</u>)



(In plane $\vec{v} = \vec{OP} = \begin{bmatrix} a \\ b \end{bmatrix}$, where



<u>OPERATIONS</u>

$+$: $\vec{v} + \vec{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x+x' \\ y+y' \\ z+z' \end{pmatrix}$

$\cdot$: $k\vec{v} = k\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} kx \\ ky \\ kz \end{pmatrix}$

$\vec{0}$: $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ → the zero vector

negative $\vec{v}$: $-\vec{v} = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$

<u>PROPERTIES:</u> $\vec{v} + \vec{w} = \vec{w} + \vec{v}$; $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$;
$\vec{0} + \vec{v} = \vec{v}$; $\vec{v} + (-\vec{v}) = \vec{0}$; $(a+b)\vec{v} = a\vec{v} + b\vec{v}$;
$a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$; $(ab)\vec{v} = a(b\vec{v})$; $1\vec{v} = \vec{v}$

(T) Let $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\vec{w} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$. <u>THEN:</u>

(1) $\vec{v} = \vec{w} \iff x = x';\ y = y';\ z = z'$

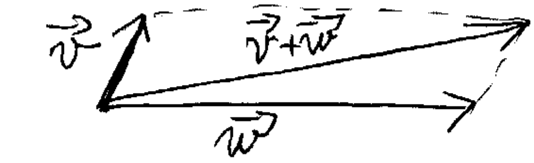
(2) $\|\vec{v}\| = \sqrt{x^2 + y^2 + z^2}$

(3) $\vec{v} = \vec{0} \iff \|\vec{v}\| = 0$

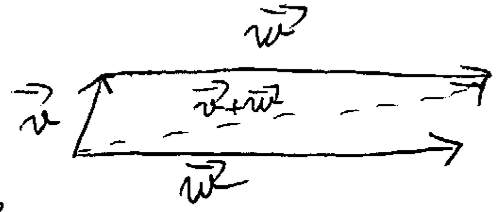
(4) $\|a\vec{v}\| = |a| \cdot \|\vec{v}\|$; a #

<u>NOTE:</u> Given $\vec{v}$ (a vector), $-\vec{v}$ has the same length, but opposite direction

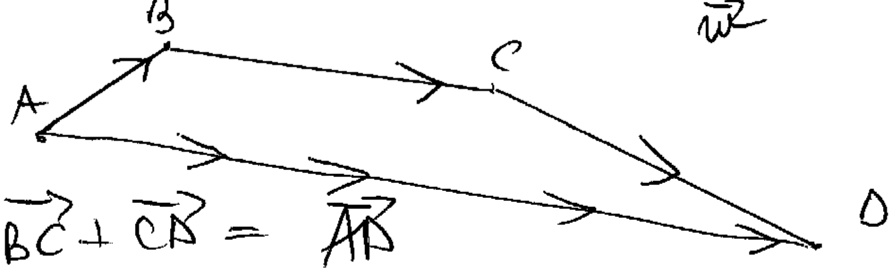
VECTOR ADDITION



- PARALLELOGRAM LAW
- TIP-to-TAIL method:

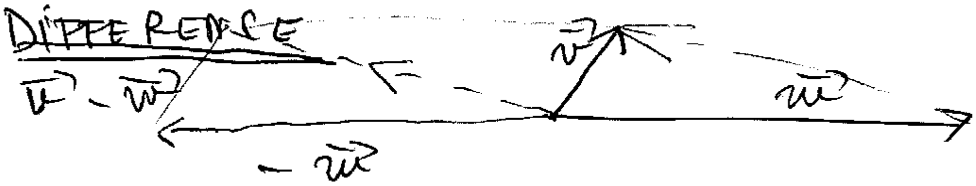


- MORE:



$$\vec{AB} + \vec{BC} + \vec{CD} = \vec{AD}$$

- DIFFERENCE



(TH) Let $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ be 2 points. THEN:

(1) $\vec{P_1P_2} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}$; (2) $d(P_1, P_2) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

(In plane: $\vec{P_1P_2} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$; $d(P_1, P_2) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$
for $P_1(x_1, y_1)$, $P_2(x_2, y_2)$).

(DEF: ||) TWO NON-ZERO vectors are parallel if they have the same or opposite direction

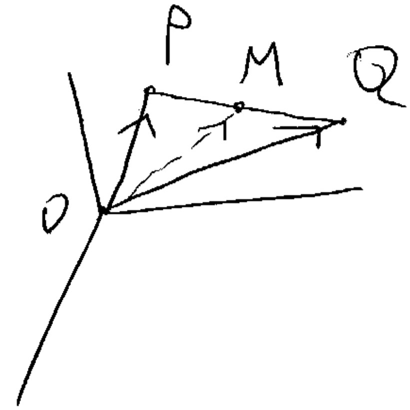
(THM) || Let $\vec{v}, \vec{w}$ be 2 vectors (nonzero)
TFAE:

(1) $\vec{v}, \vec{w}$ are parallel
(2) one of $\vec{v}, \vec{w}$ is a scalar multiple of the other.

DO: 11/141

· MIDPOINT between 2 points P, Q

$$\vec{OM} = \frac{1}{2}(\vec{OP} + \vec{OQ})$$



DO: 10/141 c) NOT

§ 3.2 DOT PRODUCT and PROJECTIONS

Def: · if $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\vec{w} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ then $\vec{v} \cdot \vec{w} = xx' + yy' + zz'$

· If $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\vec{w} = \begin{pmatrix} x' \\ y' \end{pmatrix}$ $\Rightarrow$ $\vec{v} \cdot \vec{w} = xx' + yy'$

EXP: $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = 3 + 1 + 0 = 4$

PROPERTIES: ① $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$; ② $\vec{v} \cdot \vec{w}$ is a # ③ $\vec{v} \cdot \vec{0} = 0$

④ $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$ ⑤ $(a\vec{v}) \cdot \vec{w} = a(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (a\vec{w})$

⑥ $\vec{v} \cdot (\vec{u} \pm \vec{w}) = \vec{v} \cdot \vec{u} \pm \vec{v} \cdot \vec{w}$.

Angles (Def) The angle θ $(0 \le \theta \le \pi)$ between 2 nonzero vectors u, v is satisfying:

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos\theta$$

EXP 4f/148 $\cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}$

4b/148 $\cos\theta = 0 \Rightarrow \theta = \pi/2$

<u>Def:</u> $\begin{cases} \vec{v}\cdot\vec{w} > 0 \iff \theta \text{ is acute } (0 \le \theta < \frac{\pi}{2}) \\ \vec{v}\cdot\vec{w} = 0 \iff \theta \text{ is right angle } (\theta = \pi/2) \\ \vec{v}\cdot\vec{w} < 0 \iff \theta \text{ is obtuse } (\pi/2 < \theta \le \pi) \\ \text{when } \vec{v}\cdot\vec{w} = 0 \text{ we say } \vec{v}, \vec{w} \text{ are } \underline{\text{orthogonal}}\ (\perp) \end{cases}$

$\boxed{\text{DO:}}$ 7/148 $\vec{AB} = \begin{pmatrix} -1 \\ 4 \\ 4 \end{pmatrix}$; $\vec{CD} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} \to \vec{AB}\cdot\vec{CD} = 0$

$\to \vec{AB} \perp \vec{CD} \to$ <u>Done</u>

<u>PROJECTIONS:</u> Let $\vec{v}, \vec{d} \neq \vec{0}$ be vectors.

① The projection of $\vec{v}$ onto $\vec{d}$ is given by

$$proj_{\vec{d}}(\vec{v}) = \frac{\vec{v}\cdot\vec{d}}{\|\vec{d}\|^2} \cdot d$$

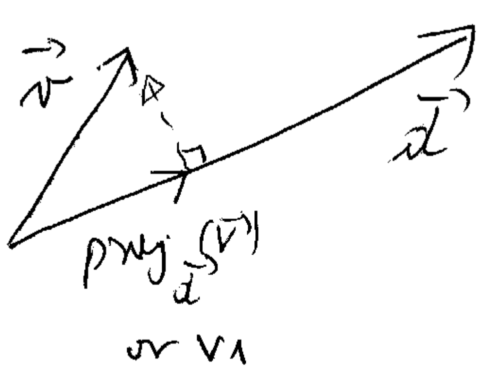
② $\vec{v} - proj_{\vec{d}}(\vec{v}) \perp \vec{d}$

③ $\vec{v}$ can be written uniquely in the form:

$\vec{v} = \vec{u}_1 + \vec{u}_2$ such that $\vec{u}_1 \parallel \vec{d}$

$\vec{u}_2 \perp d$

(In fact $\vec{u}_1 = proj_{\vec{d}}(\vec{v})$, $u_2 = \vec{v} - proj_{\vec{d}}(\vec{v})$).



DO: 10d/149

TO TRY: 106/149 ; c, f
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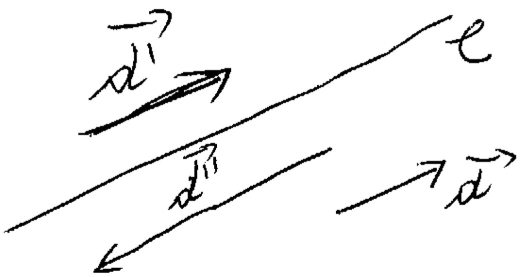
§ $\boxed{3.3}$ LINES and PLANES

§ LINES

<u>Def</u> A direction vector of a line is a non-zero vector $\vec{d}$ that is parallel to the line

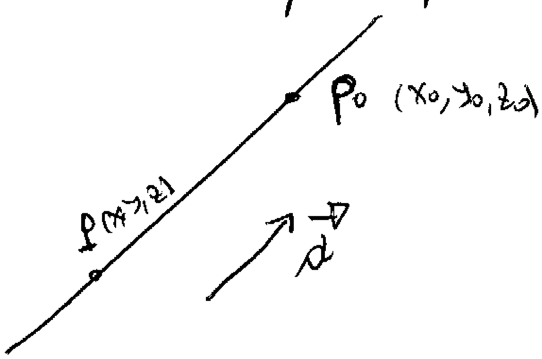


<u>DEF</u>: • <u>VECTOR EQ</u>. <u>of a line</u> that has $\vec{d}$ as a direction vector and is passing through $\vec{p_0}$:

$$\vec{p} = \vec{p_0} + t\vec{d}, \quad t \text{ scalar}$$

$$\boxed{\begin{aligned}\vec{p} &= \vec{OP}\\ \vec{p_0} &= \vec{OP_0}\end{aligned}}$$

$$\vec{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



OR: $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

• <u>SCALAR EQ of a line</u> (just <u>multiply</u> ~~/~~ identify)

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tb \end{cases}$$

Read At Home: 152–153
all that examples