University of Ottawa
Department of Mathematics and Statistics

MAT 1341D: Introduction to Linear Algebra
Instructor: Catalin Rada

Assignment 4: due April 2, 2009, 19:00 in the classroom

Family name (CAPITALS)	____________________
First name (CAPITALS)	____________________
Signature	____________________
Student number	____________________

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- The table below is for the TA. Do not write in it.
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Question	1	2	3	4	Total
Score					
Max. score	5	7	7 + 2 bonus	7	26

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Good luck! Bonne chance !

(1) (5 pts) Let $x_1, x_2, \dots x_n$ and $y_1, y_2, \dots y_n$, $n \geq 2$, be arbitrary real numbers.

(a) (4 pts) Calculate the determinant of

$$A = \begin{bmatrix} 1+x_1y_1 & 1+x_1y_2 & 1+x_1y_3 & \cdots & 1+x_1y_n \\ 1+x_2y_1 & 1+x_2y_2 & 1+x_2y_3 & \cdots & 1+x_2y_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1+x_ny_1 & 1+x_ny_2 & 1+x_ny_3 & \cdots & 1+x_ny_n \end{bmatrix}.$$

and conclude that A is not invertible if $n \geq 3$.

(b) (1 pts) Let $n = 2$. For which values of x_1, x_2, y_1 and y_2 is the matrix A not invertible?

Solution: (a) We have to show that $\det(A) = 0$ for $n > 2$. To do so we subtract the first column from the second, the third and so on until the last column. This yields

$$\det(A) = \begin{vmatrix} 1+x_1y_1 & x_1(y_2-y_1) & x_1(y_3-y_1) & \cdots & x_1(y_n-y_1) \\ 1+x_2y_1 & x_2(y_2-y_1) & x_2(y_3-y_1) & \cdots & x_2(y_n-y_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1+x_ny_1 & x_n(y_2-y_1) & x_n(y_3-y_1) & \cdots & x_n(y_n-y_1) \end{vmatrix}$$

We see that in the second column we always have a factor $y_2 - y_1$, in the third column a factor $y_3 - y_1$, etc. in the ith column a factor $y_i - y_1$). We can therefore move these factors in front of the determinant and get

$$\det(A) = (y_2-y_1)(y_3-y_1)\cdots(y_n-y_1) \begin{vmatrix} 1+x_1y_1 & x_1 & x_1 & \cdots & x_1 \\ 1+x_2y_1 & x_2 & x_2 & \cdots & x_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1+x_ny_1 & x_n & x_n & \cdots & x_n \end{vmatrix}$$

If $n > 2$ then we have at least 2 identical columns. Subtracting for example, the third column from the second yields

$$\det(A) = (y_2-y_1)(y_3-y_1)\cdots(y_n-y_1) \begin{vmatrix} 1+x_1y_1 & 0 & x_1 & \cdots & x_1 \\ 1+x_2y_1 & 0 & x_2 & \cdots & x_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1+x_ny_1 & 0 & x_n & \cdots & x_n \end{vmatrix}$$

and this is 0 by expanding along the second column.

(b) For $n = 2$ we get, using the same approach as in (a), that

$$\det(A) = (y_2-y_1) \begin{vmatrix} 1+x_1y_1 & x_1 \\ 1+x_2y_1 & x_2 \end{vmatrix} = (y_2-y_1)\big((1+x_1y_1)x_2 - (1+x_2y_1)x_1\big)$$

$$= (y_2-y_1)(x_2 + x_1y_1x_2 - x_1 - x_2y_1x_1) = (y_2-y_1)(x_2-x_1)$$

We therefore see that A is not invertible exactly when $x_1 = x_2$ or $y_1 = y_2$.

Marking: (a) The solution in (a) most be for a general $n \times n$ matrix. Doing it for $n = 3$ or $n = 4$ only gives half points.

(2) (7 pts) Let $\mathbb{S}$ be the vector space of (forward) signals, i.e., $\mathbb{S}$ consists of signals = infinite sequences $s = (s_n)_{n\in\mathbb{N}} = (s_0, s_1, \cdots)$ and that addition and scalar multiplication is done as for vectors in $\mathbb{R}^n$ (see exercise 33 in §5.2 of the textbook).

(a) (3 pts) "Blips" are special signals. They have value 1 at exactly one n and are silent otherwise. Thus, the ith blip is the signal $b^{(i)} = (b_n^{(i)})_{n\in\mathbb{N}}$ for which $b_n^{(i)} = 0$ if $i \neq n$ and $b_i^{(i)} = 1$. Show that every finite collection of blips is linearly independent. (To simplify the notation, you can show that for every natural number N the set of blips $\{b^{(1)}, b^{(2)}, \ldots, b^{(N)}\}$ is linearly independent.)

(b) (2 pts) Your receiver cuts off a signal $s = (s_n)$ at s_{1000}. Hence it acts like the function $R : \mathbb{S} \to \mathbb{R}^{1000}$ given by $R(s) = (s_n)_{0\leq n\leq 999}$. Show that R is a linear map.

(c) (2 pts) The completely lost signals are the signals $s = (s_n)$ for which $s_n = 0$ for $0 \leq n \leq 999$. Show that your receiver is loosing a lot, i.e., show that the set of completely lost signals is an infinite dimensional subspace of $\mathbb{S}$.

Solution: a) Suppose that $a_1b^{(1)} + a_2b^{(2)} + \ldots a_Nb^{(N)} = 0$. Then $a_1(1, 0, 0, 0, 0, \ldots,) + a_2(0, 1, 0, 0, 0, \ldots,) + \cdots + a_N(0, 0, 0, 0, 0, \ldots, 1, 0, 0, \ldots) = 0$, so $(a_1, a_2, \ldots, a_N, 0, 0, 0, 0, \ldots) = (0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots)$, hence $a_1 = 0, a_2 = 0, \ldots, a_N = 0$.

b) The definition of our function is $R(s_0, s_1, \ldots) = (s_0, s_1, \ldots, s_{999})$. If $t = (t_0, t_1, \ldots)$ is in $\mathbb{S}$, then $R(s + t) = R((s_0 + t_0, s_1 + t_1, \ldots)) = (s_0 + t_0, s_1 + t_1, \ldots, s_{999} + t_{999}) = (s_0, s_1, \ldots, s_{999}) + (t_0, t_1, \ldots, t_{999}) = R(s) + R(t)$ where $s = (s_0, s_1, \ldots)$. If c is a scalar, then $cs = (cs_0, cs_1, \ldots)$, so we do have $R(cs) = R((cs_0, cs_1, \ldots)) = (cs_0, cs_1, \ldots, cs_{999}) = c(s_0, s_1, \ldots, s_{999}) = cR(s)$.

c) Suppose to the contrary that H, the set of completely lost signals, is a finite dimensional subspace of $\mathbb{S}$. Then let $\infty > k = \dim H$. So any basis of H contains exactly k elements! By a) there are subsets of H that are linearly independent and arbitrarly large (from point of view of cardinality). This is a contradiction (since linearly independent subsets can be enlarged to basis of H). So, our assumption is false, therefore we proved c). Examples of linearly independent subsets of H are: $A_m = \{b^{(1000)}, b^{(1001)}, \ldots, b^{(m)}\}$, where m is any positive integer greater or equal than 1000.

(3) (7 pts) + (2 bonus pts) A function $f : \mathbb{R} \to \mathbb{R}$ is called *even* (respectively *odd*) if $f(-x) = f(x)$ (respectively $f(-x) = -f(x)$) for all $x \in \mathbb{R}$. Let $\mathbb{E}$ be the set of even functions in $\mathbb{F}[\mathbb{R}]$ and let $\mathbb{O}$ be the set of odd functions in $\mathbb{F}[\mathbb{R}]$.

(a) (2 pts) Let $T : \mathbb{F}[\mathbb{R}] \to \mathbb{F}[\mathbb{R}]$ be defined by assigning to the function $f : \mathbb{R} \to \mathbb{R}$ the function $T(f)$ defined by $(T(f))(x) = f(x) + f(-x)$ for $x \in \mathbb{R}$. This is a linear map – you do not need to prove this, but see (e) below. Find the kernel and the image of T.

(b) (1 point) Prove that $\mathbb{E}$ and $\mathbb{O}$ are subspaces of $\mathbb{F}[\mathbb{R}]$. (Hint: you can use (a).)

(c) (3 pts) We fix a positive natural number n. Find a basis of the subspace even polynomials in $\mathbb{P}_n$, and determine its dimension.

(d) (1 point) Determine the dimension of the subspace of odd polynomials in $\mathbb{P}_n$.

(e) (2 bonus pts) Show that T is a linear map.

Solution: a) $\mathrm{Ker}T = \{f \in \mathbb{F}[\mathbb{R}] | T(f) = 0 -$ the zero function $\} = \{f \in \mathbb{F}[\mathbb{R}] | T(f)(x) = 0$ for all $x \in \mathbb{R}\} = \{f \in \mathbb{F}[\mathbb{R}] | f(x) + f(-x) = 0$ for all $x \in \mathbb{R}\} = \{f \in \mathbb{F}[\mathbb{R}] | f(x) = -f(-x)$ for all $x \in \mathbb{R}\} = \mathbb{O}$ and $\mathrm{im}T = \{g \in \mathbb{F}[\mathbb{R}] | g = T(f)$ for some $f \in \mathbb{F}[\mathbb{R}]\} = \{g \in \mathbb{F}[\mathbb{R}] |$ for all x one has $g(x) = f(x) + f(-x)$ for some $f \in \mathbb{F}[\mathbb{R}]\} = \mathbb{E}$ since $g(-x) = f(-x) + f(-(-x)) = f(-x) + f(x) = g(x)$.

b) Since the image and kernel of any linear transformation are subspaces it follows by a) that $\mathbb{E}$ and $\mathbb{O}$ are subspaces of $\mathbb{F}[\mathbb{R}]$.

c) Let K be the set of all even polinomyals in $\mathbb{P}_n$. Let m be the biggest positive odd integer less or equal than n, (so m is n if n is odd; m is $n-1$ if n is even). If $p(x) = a_0 + a_1 x + a_2 x^2 + \dots a_n x^n$ is in K, then $p(x) = p(-x)$, hence $a_0 + a_1 x + a_2 x^2 + \dots a_n x^n = a_0 - a_1 x + a_2 x^2 - \dots (-1)^n a_n x^n$. It follows that $a_1 = 0, a_3 = 0, \dots, a_m = 0$. So $p(x) = a_0 + a_2 x^2 + \dots a_q x^q$, where q is the biggest positive even integer less or equal than n. We just got that $K = span\{1, x^2, \dots, x^q\}$. Since the degree of the spanning elements (polynomials) are different, we get that they form a linearly independent set! Therefore a basis (for K) is $\{1, x^2, \dots, x^q\}$. The dimension is $\frac{q}{2} + 1$.

d) Consider the following linear (see a)) transformation $T|_{\mathbb{P}_n} : \mathbb{P}_n \mapsto \mathbb{P}_n$, $T|_{\mathbb{P}_n}(f)(x) = f(x) + f(-x)$. Then $\dim \mathbb{P}_n = \dim \mathrm{Ker}(T|_{\mathbb{P}_n}) + \dim \mathrm{Ker}(T|_{\mathbb{P}_n})$, so $n + 1 = \dim \mathrm{Ker}(T|_{\mathbb{P}_n}) + \frac{q}{2} + 1$. We get by a) that the dimension of the subspace of odd polynomials in $\mathbb{P}_n$ is $n - \frac{q}{2}$.

e) Note that for all x in $\mathbb{R}$ one has that $(T(f + g))(x) = (f + g)(x) + (f + g)(-x) = f(x) + g(x) + f(-x) + g(-x) =$ (by the definition of T)
$= (f(x) + f(-x)) + (g(x) + g(-x)) = T(f)(x) + T(g)(x) = (T(f) + T(g))(x)$. So $T(f + g) = T(f) + T(g)$, were f, g are arbitrary taken from $\mathbb{F}[\mathbb{R}]$. Letting r be an arbitrary scalar, we get for all x in $\mathbb{R}$ that $(T(rf))(x) = (rf)(x) + (rf)(-x) = rf(x) + rf(-x) = r\{f(x) + f(-x)\} = r(T(f)(x)) = (rT(f))(x)$, so $T(rf) = rT(f)$ for an arbitrary f in $\mathbb{F}[\mathbb{R}]$.

(4) (7 pts) For the matrix

$$A = \begin{bmatrix} 5 & 30 & -48 \\ 3 & 14 & -24 \\ 3 & 15 & -25 \end{bmatrix}$$

(a) (3 pts) determine all eigenvalues, and
(b) (4 pts) for each eigenvalue find a basis of the corresponding eigenspace and determine its dimension.

Solution: (a) We first find the characteristic polynomial

$$c_A(x) = \det(\lambda E_3 - A) = \begin{vmatrix} \lambda-5 & -30 & 48 \\ -3 & \lambda-14 & 24 \\ -3 & -15 & \lambda+25 \end{vmatrix} = \begin{vmatrix} \lambda-5 & -30 & 48 \\ -3 & \lambda-14 & 24 \\ 0 & -1-\lambda & \lambda+1 \end{vmatrix}$$

where in the last step we subtracted row 2 from row 3. We can now pull out a factor $\lambda + 1$ from the last row and expand along the last row:

$$\begin{vmatrix} \lambda-5 & -30 & 48 \\ -3 & \lambda-14 & 24 \\ 0 & -1-\lambda & \lambda+1 \end{vmatrix} = (\lambda+1)\begin{vmatrix} \lambda-5 & -30 & 48 \\ -3 & \lambda-14 & 24 \\ 0 & -1 & 1 \end{vmatrix}$$

$$= (\lambda+1)\begin{vmatrix} \lambda-5 & 48 \\ -3 & 24 \end{vmatrix} + (\lambda+1)\begin{vmatrix} \lambda-5 & -30 \\ -3 & \lambda-14 \end{vmatrix}$$

$$= (\lambda+1)\big(24(\lambda-5) + 3\cdot 48 + (\lambda-5)(\lambda-14) - 3\cdot 30\big)$$

$$= (\lambda+1)(24\lambda - 120 + 144 + \lambda^2 - 19\lambda + 70 - 90)$$

$$= (\lambda+1)(\lambda^2 + 5\lambda + 4) = (\lambda+1)(\lambda+1)(\lambda+4)$$

The eigenvalues are the roots of $c_A(\lambda)$. Therefore they are $\lambda = -1$ and $\lambda = -4$.

(b) For $\lambda = -1$ the corresponding eigenvectors are the solutions of the homogeneous linear system $((-1)E + 3 - A)X = 0$. We row-reduce the coefficient matrix (subtract 2 row 2 from row 1, row 2 from row 1, interchange row 1 and row 2, and divide row 1 by -3)

$$-E_3 - A = \begin{bmatrix} -6 & -30 & 48 \\ -3 & -15 & 24 \\ -3 & -15 & 24 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ -3 & -15 & 24 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus the corresponding linear system is $x + 5y - 8z = 0$, i.e., $x = -5y + 8z$. The general solution with arbitrary $s, t \in \mathbb{R}$ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5s + 8t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}$$

which we have written as a linear combination of the basic solutions which are a basis of the eigenspace $E_{-1}(A)$:

$$\begin{bmatrix} -5 & 1 & 0 \end{bmatrix}^T, \quad \begin{bmatrix} 8 & 0 & 1 \end{bmatrix}$$

Its dimension is therefore $\dim E_{-1}(A) = 2$.

For $\lambda = -4$ we proceed as above. We divide all rows by -3, interchange rows 1 and 3, and then subtract multiples of rows 1 from the rows below:

$$-4E_3 - A = \begin{bmatrix} -9 & -30 & 48 \\ -3 & -18 & 24 \\ -3 & -15 & 21 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -7 \\ 1 & 6 & -8 \\ 3 & 10 & -16 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -7 \\ 0 & 1 & -1 \\ 0 & -5 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 5 & -7 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The corresponding homogeneous linear system is $x_1 = 2x_3$ and $x_2 = x_3$. Its general solution is

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \begin{bmatrix} 2s & s & s \end{bmatrix}^T = s\begin{bmatrix} 2 & 1 & 1 \end{bmatrix}^T.$$

Thus, the eigenspace $E_{-4}(A)$ has $\begin{bmatrix} 2 & 1 & 1 \end{bmatrix}^T$ as basis and is therefore 1-dimensional.

(c) **Additional problem (not required):** Decide if A is diagonalizable. If yes, give an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. If no, justify why not.

Answer: Since the multiplicity of each eigenvalue equals the number of basic eigenvectors for this eigenvalue, the matrix A is diagonalizable. The matrices P and D are:

$$P = \begin{bmatrix} 5 & 8 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

Marking: (a) 3 points,(b) 4 points (2 points for each eigenvalue).