University of Ottawa
Department of Mathematics and Statistics

MAT 1341D: Introduction to Linear Algebra
Instructor: Catalin Rada

Assignment 3: due March THU 12, 2009, 19:00 in the classroom

Family name (CAPITALS) ______________________

First name (CAPITALS) ______________________

Signature ______________________

Student number ______________________

Please read these instructions carefully:

- The table below is for the TA. Do not write in it.
- The assignment has to be submitted with the two cover pages.
- For privacy reasons, this page of the assignment will be detached, and you will only get back the remaining pages. Therefore, **fill in your name on both pages and your student number on this page only.**

Question	1	2	3	4	Total
Score					
Max. score	3	5	6	8	22

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Good luck! Bonne chance !

(1) (3 pts) In the matrix below, replace α by the **last** digit of your student number and calculate its determinant.

$$\begin{bmatrix} 1 & 1 & 1 & -2 & -1 \\ 1 & -1 & 2 & -3 & 0 \\ 1 & 3 & \alpha+1 & 3 & 3 \\ 3 & 1 & 4 & -6 & 2 \\ -1 & -1 & -1 & 2 & 2 \end{bmatrix}$$

Solution: We use elementary row operations, to produce 0's below the 1 at position (11) and then expand along the first column:

$$\begin{vmatrix} 1 & 1 & 1 & -2 & -1 \\ 1 & -1 & 2 & -3 & 0 \\ 1 & 3 & \alpha+1 & 3 & 3 \\ 3 & 1 & 4 & -6 & 2 \\ -1 & -1 & -1 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & -2 & -1 \\ 0 & -2 & 1 & -1 & 1 \\ 0 & 2 & \alpha & 5 & 4 \\ 0 & -2 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -2 & 1 & -1 & 1 \\ 2 & \alpha & 5 & 4 \\ -2 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

We expand along the last row, then pull out a -2 from the first column, and add multiples of the first row to the rows below to produce 0's below the position (11):

$$\begin{vmatrix} -2 & 1 & -1 & 1 \\ 2 & \alpha & 5 & 4 \\ -2 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -2 & 1 & -1 \\ 2 & \alpha & 5 \\ -2 & 1 & 0 \end{vmatrix} = (-2)\begin{vmatrix} 1 & 1 & -1 \\ -1 & \alpha & 5 \\ 1 & 1 & 0 \end{vmatrix} = (-2)\begin{vmatrix} 1 & 1 & -1 \\ 0 & \alpha+1 & 4 \\ 0 & 0 & 1 \end{vmatrix}$$

We now have a triangular matrix, whose determinant is the products of the entries on the diagonal:

$$(-2)\begin{vmatrix} 1 & 1 & -1 \\ 0 & \alpha+1 & 4 \\ 0 & 0 & 1 \end{vmatrix} = (-2)(\alpha+1).$$

(2) (5 pts) Find a basis for the following subspace of $\mathbb{R}^4$ and determine its dimension:

$$U = \left\{ \begin{bmatrix} a & b & c & d \end{bmatrix}^T : a + b + c + d = 0 \in \mathbb{R} \right\}$$

.

Solution: Note that an arbitrary vector in U can be rewritten as follows:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ -a-b-c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix},$$

hence

$$U = span\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \}.$$

We now check that the 3 vectors in the spanning set are linearly independent. To do so, let us solve the following equation:

$$t_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + t_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = 0.$$

We write down the corresponding augmented matrix and row-reduce:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This gives us only the trivial solution. Hence the set of vectors that span U is linearly independent, hence a basis. The dimension of U is 3.

(3) (6 pts) For the matrix

$$A = \begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix}$$

find
(a) (2 pts) the reduced row echelon form,
(b) (1 point) a basis of the row space,
(c) (1 point) a basis of the column space,
(d) (2 pts) a basis of the null space,

Solution: We row-reduce the given matrix as follows:

$$A = \begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We used the following row operations: at row 2 we added 4 times row 1, at row 3 we added 3 times row 1; and finaly at row 3 we added $\frac{-1}{3}$ times row 2. Moreover

$$\begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & 1 & 3 & -5/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 & -9/2 \\ 0 & 1 & 3 & -5/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Operations used: We scaled row 2 by $\frac{1}{-6}$, and then at row 1 we added 3 times row 2. The last matrix is the reduced row-echelon form of A. We can now solve the remaining parts.

b) Since the nonzero rows of the (reduced) row echelon form of A form a basis for the row space we get as a basis of the row spaces of A:

$$\{\begin{bmatrix} 1 & 0 & 5 & -9/2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 3 & -5/2 \end{bmatrix}\}.$$

c) Since the leading 1's of the reduced row-echelon form of A are in columns 1 and 2, a basis of the column space of A is given by columns 1 and 2 of A itself:

$$\{\begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ 7 \end{bmatrix}\}$$

d) Note that $AX = 0$ is equivalent to $RX = 0$. Hence we solve $RX = 0$ where $X = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$:

$$\begin{array}{rcrcrcl} x_1 & & + & 5x_3 & - & (9/2)x_4 & = & 0 \\ & x_2 & + & 3x_3 & - & (5/2)x_4 & = & 0 \end{array}$$

and we get: $x_3 = t$, t scalar; $x_4 = s$, s scalar; and $x_2 = -3t + \frac{5}{2}s$, $x_1 = -5t + \frac{9}{2}s$. This the general solution is

$$X = \begin{bmatrix} -5t + \frac{9}{2}s \\ -3t + \frac{5}{2}s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} -5 \\ -3 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 9/2 \\ 5/2 \\ 0 \\ 1 \end{bmatrix}$$

and the basic solutions are

$$\begin{bmatrix} -5 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 9/2 \\ 5/2 \\ 0 \\ 1 \end{bmatrix}.$$

They form a basis of the null space of A.

(4) (8 pts) In each case determine if U is a subspace of the given vector space V. Either verify all three conditions defining a subspace or give a concrete example showing that one of the conditions is not fulfilled.

(a) $V = \mathbb{M}_{3,3}$, $U = \{X \in V : X \text{ is not invertible}\}$.

Solution: U is not a subspace since U is not closed under addition: For example the following two matrices X_1 and X_2 lie in U, i.e. X_1 and X_2 are not invertible, but their sum is the 3×3-identity matrix and is therefore invertible, i.e., $X_1 + X_2 \notin U$:

$$X_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad X_1 + X_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Remark: Note that U satisfies the two other conditions of a subspace: The zero matrix lies in U, since it is not invertible, and if X is not invertible then so is sX for any scalar $s \in \mathbb{R}$. The latter can for example be seen as follows: $X \in U \iff \det(X) = 0 \implies \det(sX) = s^3 \det(X) = 0 \implies sX$ not invertible.

(b) $V = \mathbb{F}[\mathbb{R}]$ (the vector space of functions defined on $\mathbb{R}$), $U = \{f \in V : f \text{ is differentiable and } f + 2f' = \mathbf{0}\}$

Solution: This is a subspace: (1) The zero function $f = \mathbf{0}$ is clearly differentiable and its derivative is $f' = \mathbf{0}$, whence $f + 2f' = \mathbf{0} + 2\mathbf{0} = \mathbf{0}$.

(2) U is closed under scalar multiplication: For $f \in U$ and $s \in \mathbb{R}$ we know that sf is differentiable and that its derivative is $(sf)' = sf'$, whence $(sf)+2(sf)' = sf+2sf' = s(f+2f') = s\mathbf{0} = \mathbf{0}$.

(3) U is closed under addition: Suppose $f \in U$ and $g \in U$, i.e., f and g are differentiable and satisfy $f + 2f' = \mathbf{0}$ and $g + 2g' = 2\mathbf{0}$. Then the sum $f + g$ is differentiable and $(f + g)' = f' + g'$, whence $(f + g) + 2(f + g)' = f + g + 2(f' + g') = (f + 2f') + (g + 2g') = \mathbf{0} + \mathbf{0} = \mathbf{0}$.

, one for each condition.

(c) $V = \mathbb{P}_4$ and $U = \{p : p(x) = 2xq(x) \text{ for some } q \in \mathbb{P}_2\}$

Solution: 1st solution: A $q \in \mathbb{P}_2$ has the form $q(x) = a_0 + a_1x + a_2x^2$ where $a_i \in \mathbb{R}$ are arbitrary. Hence

$$U = \{2a_0x + 2a_1x^2 + 2a_2x^3 : a_i \in \mathbb{R} \text{ arbitrary}\}$$

It is now easy to check the 3 conditions for a subspace:

(1) The zero polynomial lies in U by taking all $a_i = 0$.

(2) If $p \in U$, i.e. $p(x) = 2a_0x+2a_1x^2+2a_2x^3$, $s \in \mathbb{R}$, then $sp(x) = 2(sa_0)x+2(sa_1)x^2+2(sa_2)x^3$ is a polynomial in U.

(3) If p and r are polynomials in U, say $p(x) = 2a_0x+2a_1x^2+2a_2x^3$ and $r(x) = 2b_0x+2b_1x^2+2b_2x^3$, then

$$(p + r)(x) = p(x) + r(x) = (2a_0x + 2a_1x^2 + 2a_2x^3) + (2b_0x + 2b_1x^2 + 2b_2x^3)$$
$$+ 2(a_0 + b_0)x + 2(a_1 + b_1)x^2 + 2(a_2 + b_2)x^3.$$

This shows that $p + r \in U$.

2nd solution: (1) The zero polynomial $\mathbf{0}$ lies in U since $\mathbf{0} = 2x\mathbf{0}$ and $\mathbf{0} \in \mathbb{P}_2$.

(2) Suppose $p \in U$ and $s \in R$. Thus $p(x) = 2xq(x)$ for some $q \in \mathbb{P}_2$. Therefore $(sp)(x) = sp(x) = s(2xq(x) = 2x(sq(x))$, and since $sq \in \mathbb{P}_2$, this shows $sp \in U$.

(3) Suppose p and $r \in U$, say $p(x) = 2xq_1(x)$ and $r(x) = 2xq_2(x)$ for $q_1, q_2 \in \mathbb{P}_2$. Then $(p + r)(x) = p(x) + r(x) = 2xq_1(x) + 2xq_2(x) = 2x(q_1(x) + q_2(x))$ and since $q_1 + q_2 \in U$, this shows $p + r \in U$.