University of Ottawa
Department of Mathematics and Statistics

MAT 1341D: Introduction to Linear Algebra
Instructor: Catalin Rada

Assignment 1: due Jan. 29, 2009, 19:00 in the classroom

Family name (CAPITALS)	______________
First name (CAPITALS)	______________
Signature	______________
Student number	______________

Please read these instructions carefully:

- The table below is for the TA. Do not write in it.
- The assignment has to be submitted with the two cover pages.
- For privacy reasons, this page of the assignment will be detached, and you will only get back the remaining pages. Therefore, **fill in your name on both pages and your student number on this page only.**

Question	1	2	3	4	Total
Score					
Max. score	4	4	6	8	22

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Good luck! Bonne chance !

(1) (a) (2 pts) Find the matrix A that satisfies the following equation:

$$iA - \begin{bmatrix} i & 0 & 1 \\ 1 & 1 & -i \end{bmatrix}^T = \begin{bmatrix} 0 & i \\ 3 & 2 \\ i & 1 \end{bmatrix}.$$

Solution: We get

$$iA = \begin{bmatrix} i & 0 & 1 \\ 1 & 1 & -i \end{bmatrix}^T + \begin{bmatrix} 0 & i \\ 3 & 2 \\ i & 1 \end{bmatrix} = \begin{bmatrix} i & 1 \\ 0 & 1 \\ 1 & -i \end{bmatrix} + \begin{bmatrix} 0 & i \\ 3 & 2 \\ i & 1 \end{bmatrix} = \begin{bmatrix} i & 1+i \\ 3 & 3 \\ 1+i & -i+1 \end{bmatrix}$$

Hence $A = \begin{bmatrix} 1 & 1-i \\ -3i & -3i \\ 1-i & -i-1 \end{bmatrix}$

(b) (2 pts) Find the rank of the matrix

$$\begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix}.$$

Solution: (a) $\begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix} \overset{R_2+3R_1, R_3-2R_1, R_4+4R_1}{\longrightarrow} \begin{bmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 10 & -9 \end{bmatrix} \overset{R_4-2R_2, (1/5)R_3}{\longrightarrow}$

$\begin{bmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix} \overset{R_4-5R_3, (1/5)R_2}{\longrightarrow} \begin{bmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 1 & \frac{-7}{5} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. It follows that the rank is 3.

(2) In each case either show that the statement is true or give an example with concrete numbers showing that it is false. Assume that a linear system is given with augmented matrix A and coefficient matrix C.

(a) (1 pts) If the system has a solution then $\text{rank}(A) = 1 + \text{rank}(C)$.

Solution: False. Consider

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

(b) (1 pts) If there are more variables than equations, the system is consistent.

Solution: False. Consider

$$\begin{array}{ccccccc} x & - & 3y & + & 5z & = & 7 \\ x & - & 3y & + & 5z & = & 8 \end{array}$$

(c) (1 pts) If the system is homogeneuos,has 4 equations, 6 variables, and $\text{rank}(A) = 3$, there are 3 parameters.

Solution: True. See a theorem 3 on page 19.

(d) (1 pts) If A is 6×7 and $\text{rank}(A) = 6$, the system has only the trivial solution.

Solution: False. Consider

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

(3) (6 pts) **In this problem, replace α by the last digit of your student number.** Your doctor has asked you to take every day $\alpha+8$ units of vitamin A, $17+3\alpha$ units of vitamin B and 7 units of vitamin C. There are three brands available in your local pharmacy which contain the following units of vitamins A, B, C as indicated.

	vitamin A	vitamin B	vitamin C
Brand 1	1	2	1
Brand 2	1	3	0
Brand 3	3	7	2

(a) (5 pts) Find all combination of pills that provide you with the exact daily requirement (no partial pills!).

(b) (1 pts) If all brands cost \$1, find the least expensive treatment and its cost.

Solution: (a) Let x_i be the number of pills of brand i, $i = 1, 2, 3$. Then the requirement is that the x_i satisfy the following linear system

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & 3x_3 & = & \alpha+8 \\ 2x_1 & + & 3x_2 & + & 7x_3 & = & 3\alpha+17 \\ x_1 & & & + & 2x_3 & = & 7 \end{array}$$

We write down the corresponding augmented matrix and find its reduced row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & \alpha+8 \\ 2 & 3 & 7 & 3\alpha+17 \\ 1 & 0 & 2 & 7 \end{array}\right] \sim_{(I)} \left[\begin{array}{ccc|c} 1 & 1 & 3 & \alpha+8 \\ 0 & 1 & 1 & \alpha+1 \\ 0 & -1 & -1 & -\alpha-1 \end{array}\right]$$

$$\sim_{(II)} \left[\begin{array}{ccc|c} 1 & 1 & 3 & \alpha+8 \\ 0 & 1 & 1 & \alpha+1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim_{(III)} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & 1 & \alpha+1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Elementary row operations in (I): replace rows R_2 and R_3 by $R_2 - 2R_1$ and $R_3 - R_1$; elementary row operations in (II): replace row R_3 by $R_3 + R_2$; elementary row operations in (III): replace row R_1 by $R_1 - R_2$. The new system

$$\begin{array}{rcrcrcl} x_1 & & & + & 2x_3 & = & 7 \\ & & x_2 & + & x_3 & = & \alpha+1 \end{array}$$

has infinitely many solutions, namely

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7-2t \\ \alpha+1-t \\ t \end{bmatrix}, \quad t \in \mathbb{R}.$$

The requirement that x_1, x_2 and x_3 are pills, forces

$$7-2t \geq 0, \quad \alpha+1-t \geq 0, \quad t \geq 0$$

From the second and third inequality we get $0 \leq t \leq \alpha+1$. Hence, together with the first inequality:

- If $\alpha = 0$, then $t = 0, 1$, i.e.

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \begin{bmatrix} 7 & 1 & 0 \end{bmatrix}^T \text{ or } \begin{bmatrix} 5 & 0 & 1 \end{bmatrix}^T$$

- If $\alpha = 1$ then $t = 0, 1, 2$, i.e.

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \begin{bmatrix} 7 & 2 & 0 \end{bmatrix}^T, \begin{bmatrix} 5 & 1 & 1 \end{bmatrix}^T, \begin{bmatrix} 3 & 0 & 2 \end{bmatrix}^T$$

- If $\alpha \geq 2$ then $t = 0, 1, 2, 3$, i.e.,

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \begin{bmatrix} 7 & \alpha+1 & 0 \end{bmatrix}^T, \begin{bmatrix} 5 & \alpha & 1 \end{bmatrix}^T, \begin{bmatrix} 3 & \alpha-1 & 2 \end{bmatrix}^T, \begin{bmatrix} 1 & \alpha-2 & 3 \end{bmatrix}^T.$$

(b) The cost of the treatment in \$ is $x_1 + x_2 + x_3 = 8 + \alpha - 2t$. It is least expensive for t maximal, i.e.,

- If $\alpha = 0$, then $t = 1$, and the cost is \$6.
- If $\alpha = 1$ then $t = 2$, and the cost is \$5.
- If $\alpha \geq 2$ then $t = 3$, and the cost is $\$2 + \alpha$.

(4) (6+2 pts) Consider the following system of linear equations:

$$\begin{array}{ccccccc} x & - & 3y & + & 5z & = & 7 \\ x & + & (a-3)y & + & 7z & = & 8 \\ x & - & 3y & + & (a^2+2a+5)z & = & a+7 \end{array}$$

(a) Determine the values of a for which the system has:
  (i) no solution
  (ii) infinitely many solutions
  (iii) exactly one solution.

(b) In case (ii), describe the solution set of the system.

Solution: The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -3 & 5 & 7 \\ 1 & a-3 & 7 & 15 \\ 1 & -3 & a^2+2a+5 & a+7 \end{array}\right)$$

We perform the following operations: $L_s \rightsquigarrow -L_1 + L_2$ and $L_3 \rightsquigarrow -L_1 + L_3$. This yields

$$M = \left(\begin{array}{ccc|c} 1 & -3 & 5 & 7 \\ 0 & a & 2 & 8 \\ 0 & 0 & a(a+2) & a \end{array}\right).$$

Thus, we obtain:

- If $a = -2$ the last row of M is $\left(\begin{array}{ccc|c} 0 & 0 & 0 & -2 \end{array}\right)$. Hence the system is not solvable.
- If $a = 0$ then $M = \left(\begin{array}{ccc|c} 1 & -3 & 5 & 7 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right)$ and the reduced row-echelon form of M is
$$R = \left(\begin{array}{ccc|c} 1 & -3 & 0 & -13 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right).$$
Hence the system has infinitely many solutions.
- If $a \notin \{-2, 0\}$ then $M = \left(\begin{array}{ccc|c} 1 & -3 & 5 & 7 \\ 0 & * & 2 & 8 \\ 0 & 0 & * & a \end{array}\right)$ where the stars "$*$" are non-zero numbers. It follows that the system has a unique solution.

The answer to (a) is therefore
(i) The system is not solvable if $a = -2$.
(ii) The system has infinitely many solutions if $a = 0$.
(iii) The system is uniquely solvable if $a \notin \{0, -2\}$.

To answer (b) we use the matrix R above. The leading (= non-free) variables are x and z, the free variable is y. We put $y = t$ ($t \in \mathbb{R}$). Then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3t-13 \\ t \\ 4 \end{pmatrix} = \begin{pmatrix} -13 \\ 0 \\ 4 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad (t \in \mathbb{R})$$

are all solutions.