MAT 1339 A   Fall 2010   October 14th, 11:30   Prof. C. Rada

TEST #1

Max = 20

Student Number: ______________________________

• Time: 80 min.

• Only basic scientific calculators are permitted: non-programmable, non-graphing, no differentiation or integration capability. Notes or books are not permitted.

• Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.

• Write *only* in non-erasable ink (ball-point or pen), not in pencil. Cross out, if necessary, but do not erase or overwrite. Graphs and sketches may be drawn in pencil.

• Problems require complete and clearly presented solutions and carry part marks if there is substantial correct work toward the solution.

1. [2 points]   Find two integers with sum 40 and whose product is a maximum.

Solution 1: Let x and y be two integers such that $x + y = 40$. The product xy of two integers is $x(40 - x)$ by substitution. Let $f(x) = x(40 - x)$. This function has a maximum at the critical point, because the coefficient of the term x^2 of $f(x)$ is negative. As $f'(x) = 40 - 2x$, $x = 20$ is the critical point and hence $x = y = 20$ give a maximum of the product of x and y.

Solution 2: As the product of a negative number and a positive number is negative, we may assume that two integers x and y are positive (of course, sum of negative numbers cannot be 40) such that $x + y = 40$. As

$$\frac{x+y}{2} \geq \sqrt{xy} \text{ and the equality holds if and only if } x = y, \tag{1}$$

we have $x = y = 20$ to maximize their product.

The inequality (**??**) is true, because $\frac{x+y}{2} \geq \sqrt{xy} \iff (\frac{x+y}{2})^2 \geq (\sqrt{xy})^2 \iff (\frac{x-y}{2})^2 \geq 0$ and the equality of the last inequality holds if and only if $x = y$.

2. [2 points]   Assume that $f(x) = g(h(x))$ and that $h(7) = 19$, $h'(7) = -1$, $g(19) = -1$, $g'(19) = 1$. Find $f'(7)$.

Solution: By the chain rule, we have $f'(x) = g'(h(x)) \cdot h'(x)$, hence

$$f'(7) = g'(h(7)) \cdot h'(7) = g'(19) \cdot (-1) = 1 \cdot (-1) = -1.$$

3. [2 points]   Find the equation of the tangent line to the graph of $f(x) = x^3 + x^2 + x - 1$ at $(1, 2) = (1, f(1))$.

Solution: As $f'(x) = 3x^2 + 2x + 1$, the slope of the tangent line at $x = 1$ is

$$f'(1) = 3 \cdot 1^2 + 2 \cdot 1 + 1 = 6.$$

Let $y = 6x + n$ be the tangent line for some constant n. As the tangent line passes through $(1, 2)$, we get $2 = 6 \cdot 1 + n$, so $n = -4$. Therefore, the equation of the tangent line is $y = 6x - 4$.

4. [7 points] **Graph** the function $f(x) = \frac{x^2+x-2}{x^2}$ by following the steps:

— Calculate $f'(x)$ and find critical numbers. When is f increasing? When is f decreasing? Find local maximums and local minimums (if any). Find absolute maximums and absolute minimums (if any).

— Calculate $f''(x)$ and find inflection points. When is f concave up? When is f concave down?

— Find the vertical and horizontal asymptotes (if any).

Solution:

$$f'(x) = \frac{(x^2+x-2)'x^2 - (x^2+x-2)(x^2)'}{x^4} = \frac{(2x+1)x^2 - (x^2+x-2)\cdot 2x}{x^4} = \frac{-x(x-4)}{x^4}.$$

As the denominator x^4 cannot be 0, $f'(x) = 0$ has the root $x = 4$, which is the critical point.

$$f''(x) = \frac{(-x^2+4x)'(x^4) - (-x^2+4x)(x^4)'}{(x^4)^2} = \frac{(-2x+4)x^4 - (-x^2+4x)(4x^3)}{x^8} = \frac{2x^4(x-6)}{x^8}.$$

As the denominator x^8 cannot be 0, $f''(x) = 0$ has the root $x = 6$, which is a candidate for the inflection point. As

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}(1 + \frac{1}{x} - \frac{2}{x^2}) = \lim_{x\to 0}(1 + \frac{1}{x}(1 - \frac{2}{x})) = -\infty,$$

$x = 0$ (or y-axis) is the vertical asymptote. As

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}(1 + \frac{1}{x} - \frac{2}{x^2}) = 1,$$

$y = 1$ (yellow line in the picture below) is the horizontal asymptote.

We divide the real line into intervals with respect to $x = 0$, $x = 4$ and $x = 6$. Then we have the following table:

Interval	$(-\infty,0)$	(0,4)	(4,6)	$(6,\infty)$
Sign of $f'(x)$	$-$	$+$	$-$	$-$
Sign of $f''(x)$	$-$	$-$	$-$	$+$
Shape of curve	DU(Red)	IU(Blue)	DU(Gray)	DD(Green)

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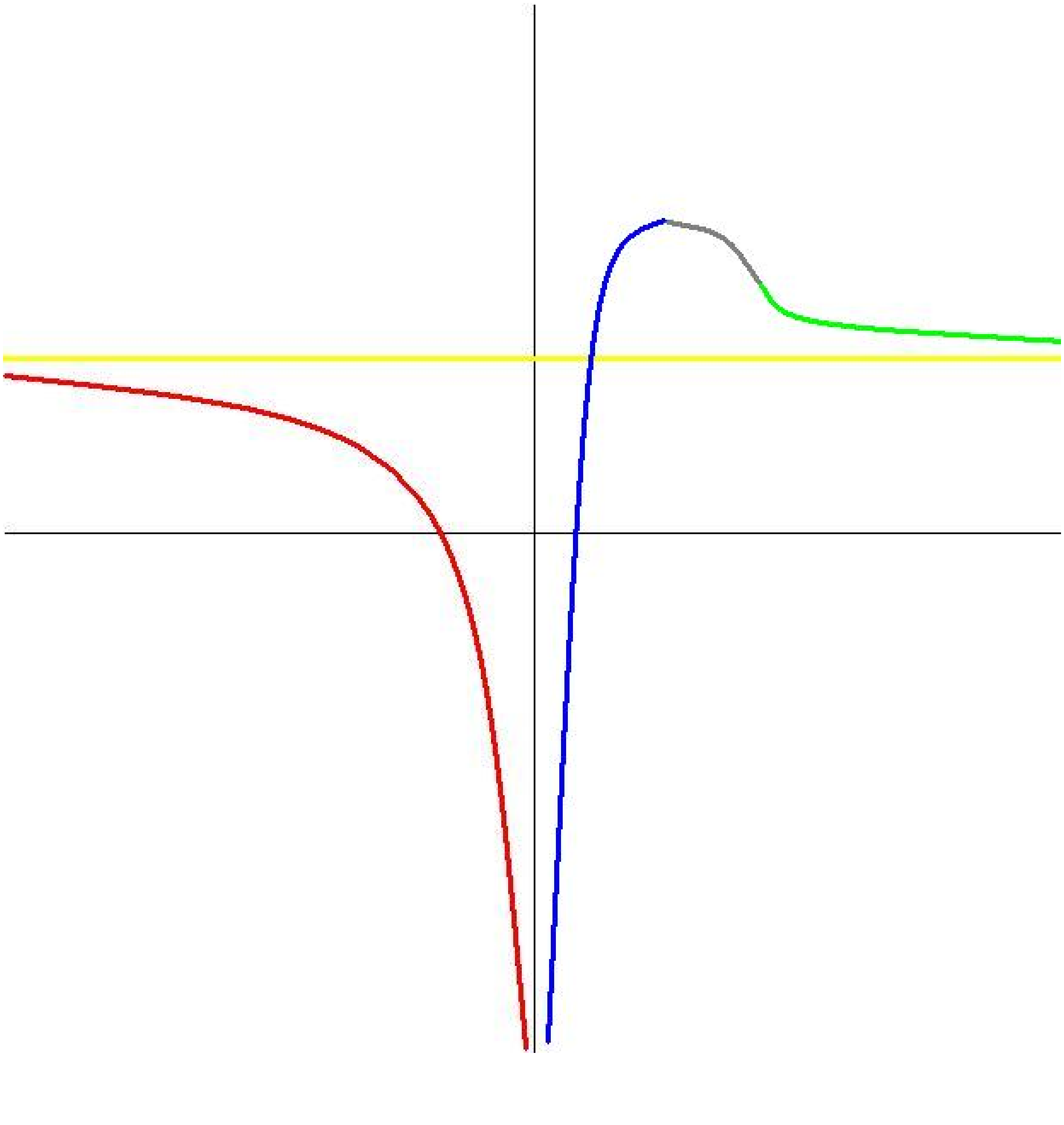
where DU:=Decreasing and concave Up, IU:=Increasing and concave Up, and DD:=Decreasing and concave Down.

According to the table or the graph, $x = 6$ is the inflection point, $(0, 4)$ is the interval of increasing ($\mathbb{R}\backslash(0, 4)$ is the union of the intervals of decreasing), $(6, \infty)$ is the interval of concave up ($\mathbb{R}\backslash(6, \infty)$ is the union of the intervals of concave down), and $(4, f(4)) = (4, 9/8)$ is both local maximum and absolute maximum. There are no local minimum and absolute minimum for this this function.

Graph of $f(x)$:

As $f(x) = 0$ has roots $x = -2$ and $x = 1$, x-intercepts are $(-2, 0)$ and $(1, 0)$.



5. [3 points]   Use the definition of the derivative to find $f'(x)$ if $f(x) = -x^2 - x^3$. And then use the Power Rule to verify your answer.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{-(x+h)^2 - (x+h)^3 - [-x^2 - x^3]}{h} \\ &= \lim_{h \to 0} \frac{-2xh - h^2 - 3x^2h - 3xh^2 - h^3}{h} \\ &= \lim_{h \to 0} -2x - h - 3x^2 - 3xh - h^2 \\ &= -2x - 3x^2. \end{aligned}$$

By the power rule, $f'(x) = -2x - 3x^2$.

6. [2 points] Differentiate the following function:

$h(x) = (x^3 + x^2 + x + 1)^{\frac{1}{2010}}$

Solution: By the chain rule, we have

$$\begin{aligned} h'(x) &= \frac{1}{2010}(x^3 + x^2 + x + 1)^{\frac{1}{2010}-1} \cdot (x^3 + x^2 + x + 1)' \\ &= \frac{1}{2010}(x^3 + x^2 + x + 1)^{-\frac{2009}{2010}} \cdot (3x^2 + 2x + 1). \end{aligned}$$

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• Problems require complete and clearly presented solutions and carry part marks if there is substantial correct work toward the solution.

1. [2 points] Find two integers with sum 60 and whose product is a maximum.

Solution 1: Let x and y be two integers such that $x + y = 60$. The product xy of two integers is $x(60 - x)$ by substitution. Let $f(x) = x(60 - x)$. This function has a maximum at the critical point, because the coefficient of the term x^2 of $f(x)$ is negative. As $f'(x) = 60 - 2x$, $x = 30$ is the critical point and hence $x = y = 30$ give a maximum of the product of x and y.

Solution 2: As the product of a negative number and a positive number is negative, we may assume that two integers x and y are positive (of course, sum of negative numbers cannot be 60) such that $x + y = 60$. As

$$\frac{x+y}{2} \geq \sqrt{xy} \text{ and the equality holds if and only if } x = y, \tag{2}$$

we have $x = y = 30$ to maximize their product.

The inequality (**??**) is true, because $\frac{x+y}{2} \geq \sqrt{xy} \iff (\frac{x+y}{2})^2 \geq (\sqrt{xy})^2 \iff (\frac{x-y}{2})^2 \geq 0$ and the equality of the last inequality holds if and only if $x = y$.

2. [2 points] Assume that $f(x) = g(h(x))$ and that $h(17) = 3$, $h'(17) = -1$, $g(3) = -1$, $g'(3) = 1$. Find $f'(17)$.

Solution: By the chain rule, we have $f'(x) = g'(h(x)) \cdot h'(x)$, hence

$$f'(17) = g'(h(17)) \cdot h'(17) = g'(3) \cdot (-1) = 1 \cdot (-1) = -1.$$

3. [2 points] Find the equation of the tangent line to the graph of $f(x) = x^3 - x^2 - x - 1$ at $(1, -2) = (1, f(1))$.

Solution: As $f'(x) = 3x^2 - 2x - 1$, the slope of the tangent line at $x = 1$ is

$$f'(1) = 3 \cdot 1^2 - 2 \cdot 1 - 1 = 0.$$

Let $y = n$ be the tangent line for some constant n. As the tangent line passes through $(1, -2)$, we get $n = -2$. Therefore, the equation of the tangent line is $y = -2$.

4. [7 points] **Graph** the function $f(x) = \frac{-x^2-x+2}{x^2}$ by following the steps:

— Calculate $f'(x)$ and find critical numbers. When is f increasing? When is f decreasing? Find local maximums and local minimums (if any). Find absolute maximums and absolute minimums (if any).

— Calculate $f''(x)$ and find inflection points. When is f concave up? When is f concave down?

— Find the vertical and horizontal asymptotes (if any).

Solution:

$$f'(x) = \frac{(-x^2-x+2)'x^2 - (-x^2-x+2)(x^2)'}{x^4} = \frac{(-2x-1)x^2 - (-x^2-x+2)\cdot 2x}{x^4} = \frac{x(x-4)}{x^4}.$$

As the denominator x^4 cannot be 0, $f'(x) = 0$ has the root $x = 4$, which is the critical point.

$$f''(x) = \frac{(x^2-4x)'(x^4) - (x^2-4x)(x^4)'}{(x^4)^2} = \frac{(2x-4)x^4 - (x^2-4x)(4x^3)}{x^8} = \frac{-2x^4(x-6)}{x^8}.$$

As the denominator x^8 cannot be 0, $f''(x) = 0$ has the root $x = 6$, which is a candidate for the inflection point. As

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}(-1 - \frac{1}{x} + \frac{2}{x^2}) = \lim_{x\to 0}(-1 + \frac{1}{x}(\frac{2}{x} - 1)) = \infty,$$

$x = 0$ (or y-axis) is the vertical asymptote. As

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}(-1 - \frac{1}{x} + \frac{2}{x^2}) = -1,$$

$y = -1$ (yellow line in the picture below) is the horizontal asymptote.

We divide the real line into intervals with respect to $x = 0$, $x = 4$ and $x = 6$. Then we have the following table:

Interval	$(-\infty,0)$	(0,4)	(4,6)	$(6,\infty)$
Sign of $f'(x)$	+	−	+	+
Sign of $f''(x)$	+	+	+	−
Shape of curve	IU(Red)	DU(Blue)	IU(Gray)	ID(Green)

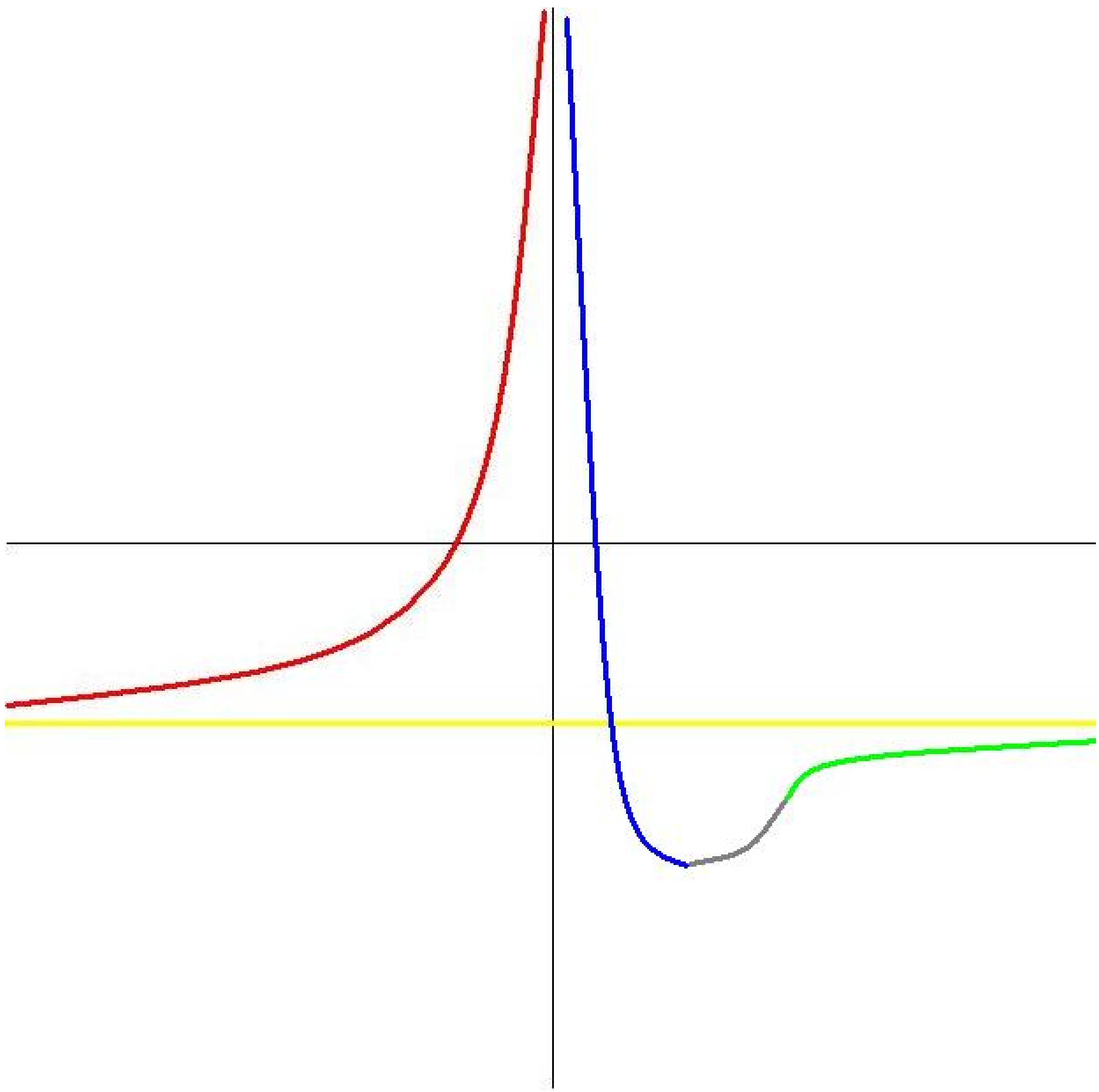
,

where IU:=Increasing and concave Up, DU:=Decreasing and concave Up, and ID:=Increasing and concave Down.

According to the table or the graph, $x = 6$ is the inflection point, $(0, 4)$ is the interval of decreasing ($\mathbb{R}\backslash(0, 4)$ is the union of the intervals of increasing), $(6, \infty)$ is the interval of concave down ($\mathbb{R}\backslash(6, \infty)$ is the union of the intervals of concave up), and $(4, f(4)) = (4, -9/8)$ is both local minimum and absolute minimum. There are no local maximum and absolute maximum for this this function.

Graph of $f(x)$:

As $f(x) = 0$ has roots $x = -2$ and $x = 1$, x-intercepts are $(-2, 0)$ and $(1, 0)$.



5. [3 points] Use the definition of the derivative to find $f'(x)$ if $f(x) = x^3 - x$. And then use the Power Rule to verify your answer.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^3 - (x+h) - [x^3 - x]}{h} \\ &= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} \\ &= \lim_{h \to 0} 3x^2 + 3xh + h^2 - 1 \\ &= 3x^2 - 1. \end{aligned}$$

By the power rule, $f'(x) = 3x^2 - 1$.

6. [2 points] Differentiate the following function:

$h(x) = (x^3 + x^2 - x - 1)^{\frac{1}{2011}}$

Solution: By the chain rule, we have

$$\begin{aligned} h'(x) &= \frac{1}{2011}(x^3 + x^2 - x - 1)^{\frac{1}{2011}-1} \cdot (x^3 + x^2 - x - 1)' \\ &= \frac{1}{2011}(x^3 + x^2 - x - 1)^{-\frac{2010}{2011}} \cdot (3x^2 + 2x - 1). \end{aligned}$$