

These are only some comments. You must come to the lectures to see solutions, graphs, how we apply formulae, etc. It may contain errors :).

Comments on lecture 1

1.1 Rates of change and the slope of a curve

Think about the following

Example 0.0.1.

<i>Time t (min)</i>	<i>Number N of students in a stadium</i>
0	10
2	24
4	44
6	55
8	63
10	76

Average rate of change from $t_1 = 8$ to $t_2 = 10$ is $\frac{N_2 - N_1}{t_2 - t_1} = \frac{76 - 63}{10 - 8} = 6.5$ students per minute (a positive number !!!)

Example 0.0.2.

<i>Time t (min)</i>	<i>Number N of students in a classroom</i>
0	100
2	91
4	49
6	30
8	21
10	0

Average rate of change from $t_1 = 4$ to $t_2 = 8$ is $\frac{N_2 - N_1}{t_2 - t_1} = \frac{21 - 49}{8 - 4} = -7$ students per minute (a negative number !!!)

GRAPH YOURSELF THE DATA GIVEN IN THE ABOVE EXAMPLES! AND NOTE THAT:

Positive rate of change suggests increasing.

Negative rate of change suggests decreasing.

Definition 0.0.1. Given a curve, a secant line is a line that connects two points on the same curve.

Graph a curve and some secants!

Definition 0.0.2. Slope of secant line: $\frac{y_2 - y_1}{x_2 - x_1} = \text{AVERAGE RATE OF CHANGE}$.

Do example 0.0.2 between 4 and 10: $\frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 49}{10 - 4} = \frac{-49}{6} = -8.1666 = \text{AVERAGE RATE OF CHANGE}$.

1.2 Instantaneous Rate of Change

Look at example 0.0.1.

Instantaneous rate of change at $t = 8$: estimate!

— by the average rate of change over the smallest interval:

$$\frac{63 - 55}{8 - 6} = 4 \text{ or } \frac{76 - 63}{10 - 8} = 6.5$$

— by the slope of tangent line $\frac{y_2 - y_1}{x_2 - x_1} = \frac{63 - y_1}{8 - x_1} =$

Summary for: Rates of Change and Slope of Lines

- Average rate of change \Leftrightarrow Slope of secant line.
- Instantaneous rate of change \Leftrightarrow Slope of tangent line.

Rates of Change Using Equations

Let $y = f(x)$ be a function, and $(a, f(a)), (b, f(b))$ be two points on the graph of $y = f(x)$, with $a \neq b$. (Draw a nice graph!) Then:

$$\text{Slope of secant line} = \frac{f(b)-f(a)}{b-a}.$$

Another way of getting the slope of a secant line is: $\frac{f(a+h)-f(a)}{h}$. Draw another graph now!

The Difference Quotient

The difference quotient is given by: $\frac{f(a+h)-f(a)}{h}$.

Question: What happens as h becomes smaller?

We find out by computing some difference quotients and by drawing the corresponding secants! Draw yourself a nice big graph, fix an a , then draw secants as h becomes smaller!!!

Answer: As h becomes **smaller**, the slope of the **secant** line becomes an increasingly **closer** estimate of the slope of the **tangent** line.

Example 0.0.3. Suppose that $y = f(t) = 3 - 4.9t^2$.

$$\begin{aligned} \text{— Compute the difference quotient as follows: } \frac{f(a+h)-f(a)}{h} &= \frac{[3-4.9(a+h)^2]-[3-4.9a^2]}{h} = \\ \frac{3-4.9(a^2+2ah+h^2)-3+4.9a^2}{h} &= \frac{-9.8ah-4.9h^2}{h} = -9.8a - 4.9h. \end{aligned}$$

— Compute the average rate of change from $t_1 = 1$ to $t_2 = 3$.

So $a = 1$, $h = 3 - 1 = 2$. Thus $\frac{f(a+h)-f(a)}{h} = -19.6$.

— Give an estimate of the instantaneous rate of change at $t = 1$ as follows:

let $a = 1$, $h = 0.01$. We get $\frac{f(a+h)-f(a)}{h} = -9.8 \times 1 - 4.9 \times (0.01) = -9.795$.

If we want a better estimate we choose a smaller h .

Summary of The Difference Quotient:

- The difference quotient \Leftrightarrow Slope of secant line
- The difference quotient generates an increasingly accurate estimate of slope of tangent line as h becomes closer to 0.

— In other words: $\lim_{h \rightarrow 0} \frac{f(a+h)-f(a)}{h} = \text{slope of tangent line!!!}$

Comments on lecture 2**1.3 LIMITS**

We shall discuss limits of sequences and limits of functions!

Definition 0.0.3. A sequence is an ordered list of real numbers, denoted by $a_1, a_2, a_3, a_4, \dots$, or $\{a_n\}_{n=1}^{\infty}$

Example 0.0.4. 10, -3, 23, 67, 34.78, 78, 2010, 0, -1,

Note that $a_1 = 10, a_2 = -3, a_3 = 23, a_4 = 67, a_7 = 2010, a_8 = 0$,

Some sequences are defined by relations:

Example 0.0.5. $a_1 = 20, a_2 = 10, a_n = 3a_{n-1} + 2a_{n-2}$ for $n \geq 3$.

Let us compute the first terms: $a_3 = 3 \times a_2 + 2 \times a_1 = 3 \times 10 + 2 \times 20 = 70$, $a_4 = 3 \times a_3 + 2 \times a_2 = 3 \times 70 + 2 \times 10 = 230$.

Look now at the

Example 0.0.6. Define $a_n = \frac{n}{n+1}$.

Note that $a_1 = \frac{1}{2}, a_2 = \frac{2}{3}, a_3 = \frac{3}{4}, a_4 = \frac{4}{5}, a_5 = \frac{5}{6}$

Intuition tells us that as n increases (becomes very big), a_n gets as close as we want to the number 1.

The formal

Definition 0.0.4. We write $\lim_{n \rightarrow \infty} a_n = L$ if for every positive number $\epsilon > 0$, there is a positive integer $n_0 > 0$ such that for all $n > n_0$ one has that

$$|a_n - L| < \epsilon$$

Terminology: — 'as n becomes very big' $n > n_0$;

— ' a_n gets as close as we want to L ' $|a_n - L| < \epsilon$

Question: when does the limit of a sequence exist?

Some answers are:

— when the sequence is increasing (except the first couple of terms) and bounded above by a number;

— when the sequence is decreasing (except the first couple of terms) and bounded below by a number;

Think of the following: $a_n = \frac{n}{n+1}, b_n = \frac{2010}{7^n}$ for $n \geq 1$.

Another

Example 0.0.7. $a_n = \begin{cases} \frac{(-1)^{n/2}}{3^n}, & \text{if } n \text{ is even} \\ \frac{1}{n}, & \text{if } n \text{ is odd} \end{cases}$

Find its limit!

Recall

Definition 0.0.5. A function $f : A \mapsto B$ is a rule that assigns to each element x in A a unique element $f(x)$ in B . The set A is called the domain of f . The set of all possible values of $f(x)$ is called the range of f .

Example 0.0.8. $y = f(x) = \sqrt{7 - x^2}, g(x) = 2x + 1, h(x) = \frac{x^2 - x}{23x - 1}$

Consider the following

Example 0.0.9. $f(x) = \frac{2x}{x^2 + 1}$ AS x approaches 2. Do some computations: $f(1.9) =, f(1.99) =, f(1.999) =, f(2.1) =, f(2.01) =, f(2.001) = \dots$

Intuition tells us that as x approaches 2, $f(x)$ gets as close as we want to 0.8.

The formal

Definition 0.0.6. We write $\lim_{x \rightarrow a} f(x) = L$ if for every positive real number $\epsilon > 0$, there exists a positive real number $\delta > 0$ such that for all $x \neq a$ with $|x - a| < \delta$ one has that $|f(x) - L| < \epsilon$.

Thus: $\lim_{x \rightarrow 2} \frac{2x}{x^2 + 1} = 0.8$.

Note that $|x - a| < \delta$ means 'as x approaches a '; and that $|f(x) - L| < \epsilon$ means that ' $f(x)$ gets as close as we want to L '.

Left-Hand Limit and Right-Hand Limit Consider the same function $f(x) = \frac{2x}{x^2+1}$, and note that

$$\text{As } x \text{ approaches 2 from the left side } \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} \frac{2x}{x^2+1} = 0.8,$$

$$\text{As } x \text{ approaches 2 from the right side } \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} \frac{2x}{x^2+1} = 0.8.$$

They are called the Left-Hand Limit and Right-Hand Limit!

Question: when does the limit of a function exist?

Answer: If:

$$- \lim_{x \rightarrow a^-} f(x) \text{ exists AND}$$

$$- \lim_{x \rightarrow a^+} f(x) \text{ exists AND}$$

$$- \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x).$$

Question: when the limit of a function does not exist?

Answer(s):

$$- \text{If } \lim_{x \rightarrow a^-} f(x) \text{ does not exist OR}$$

$$- \lim_{x \rightarrow a^+} f(x) \text{ does not exist OR}$$

$$- \lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x).$$

Evaluate the Limits of Functions: Using Properties

Proposition 0.0.1. $\lim_{x \rightarrow a} c = c$

$$\lim_{x \rightarrow a} x = a$$

$$\lim_{x \rightarrow a} \{f(x) + g(x)\} = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

$$\lim_{x \rightarrow a} \{f(x) - g(x)\} = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$$

$$\lim_{x \rightarrow a} cf(x) = c \lim_{x \rightarrow a} f(x)$$

$$\lim_{x \rightarrow a} f(x)g(x) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$$

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \text{ provided } \lim_{x \rightarrow a} g(x) \neq 0$$

$$\lim_{x \rightarrow a} f(x)^n = \{\lim_{x \rightarrow a} f(x)\}^n, \text{ where } n \text{ is a rational number.}$$

$$\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} \text{ if the roots make sense!}$$

Compute $\lim_{x \rightarrow 1} (3 - 5x)(x^2 - 2010)$; $\lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{x^2 - 6x + 5}$; $\lim_{x \rightarrow 0} \frac{\sqrt{9+x} - 3}{x}$ (for this one you can think about rationalizing!);

1.4 Continuity and Limits

A function $f(x)$ is called continuous at the point a if: $f(a)$ is defined; $\lim_{x \rightarrow a} f(x) = f(a)$.

Is the function $f(x) = \frac{3x^2}{x^3+1}$ continuous at 0?

Discontinuity

A function $f(x)$ is discontinuous at a if either $f(a)$ is not defined; OR $\lim_{x \rightarrow a} f(x)$ does NOT exist; OR $\lim_{x \rightarrow a} f(x) \neq f(a)$.

Examples include:

Example 0.0.10. $f(x) = \begin{cases} x^2 & \text{if } x < 1, \\ 56 & \text{if } x = 1, \\ 2 - x & \text{if } x > 1. \end{cases}$

It is called *Removable Discontinuity* because $\lim_{x \rightarrow a} f(x) \neq f(a)$

Example 0.0.11. $f(x) = \begin{cases} x^2 & \text{if } x < 1, \\ 56 & \text{if } x = 1, \\ 2010 - x & \text{if } x > 1. \end{cases}$

It is called *Jump Discontinuity* because $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$.

Example 0.0.12. $f(x) = \frac{x-2010}{x-2}$

It is called *Essential Discontinuity* because $\lim_{x \rightarrow a^-} f(x)$, $\lim_{x \rightarrow a^+} f(x)$ do not exist! (They are NOT finite.)

Comments on lecture 3

1.5 Intro to Derivatives

— The **Derivative** of a function $y = f(x)$ is a new function (created from the given function $y = f(x)$); it can be denoted by $f'(x)$ or y' or $\frac{dy}{dx}$ or $\frac{df}{dx}$;

— It represents the **slope** of the tangent line to the given curve ($y = f(x)$) at the point $(x, f(x))$, or the **instantaneous rate of change** at x ;

— **The Differentiation** is the process (operation) of finding a derivative;

— OK, but what is it?

Definition 0.0.7. The derivative of a function $f(x)$ is the new function $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$, if the limit exists.

Thus, the derivative of a function $f(x)$ at $x = a$ is $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$, if the limit exists.

Definition 0.0.8. If the limit exists, we say f is differentiable at $x = a$.

Example 0.0.13. If $f(x) = x$ find $f'(x)$ using the definition.

What are the graphs of f and f' ? Is it what you expect?

Example 0.0.14. If $f(x) = x^2$ find $f'(x)$ using the definition.

What are the graphs of f and f' ?

Example 0.0.15. If $f(x) = x^3$ find $f'(x)$ using the definition.

What are the graphs of f and f' ?

Example 0.0.16. If $f(x) = \frac{1}{x}$ find $f'(x)$ using the definition.

So, we can generalize:

Theorem 0.0.1. (The Power Rule) Suppose that $f(x) = x^n$. Then $f'(x) = nx^{n-1}$.

Example 0.0.17. If $y = f(t) = -5t^2 + 6t + 2010$ find the instantaneous rate of change at $t = 3$

Solution: Compute first $f'(t)$ WITH THE HELP OF THE DEFINITION, and then plug in $t = 3$ to get $f'(3)$. (We used that the instantaneous rate of change at $t = 3$ is in fact the derivative of f computed at $t = 3$.)

Now note:

Theorem 0.0.2. (*Differentiability implies Continuity*) If f is differentiable at a , then f is continuous at a .

But Continuity **does not imply** Differentiability. Look at the function $f(x) = |x|$, and $a = 0$. The function f is continuous at 0, but it is **not differentiable** at 0 because the left and right hand limits are not equal!

Question: Where functions Are not Differentiable at a point?

Answers: — when the function is not continuous at that point;

— when there is a cusp (sharp point/vertex);

— when there is a vertical tangent to the graph of the function at that point.

2.1 Derivative of a polynomial

Theorem 0.0.3. (*Derivative of a constant function*) If $f(x) = c$, where c is a number, then $f'(x) = 0$.

Theorem 0.0.4. (*Sum Rule and Difference Rule*) If $h(x) = f(x) + g(x)$, and $k(x) = f(x) - g(x)$, then $h'(x) = f'(x) + g'(x)$ and $k'(x) = f'(x) - g'(x)$.

Theorem 0.0.5. (*Constant Multiple Rule*) If $h(x) = cf(x)$, where c is a number, then $h'(x) = cf'(x)$.

SO, FROM NOW ON YOU MAY COMPUTE THE DERIVATIVE OF ANY POLYNOMIAL!

Do: 5a/84; 13a/84; 4e/83

Comments on lecture 4

2.2 Product Rule

Theorem 0.0.6. One has that $\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x)$. In the other notation one has that $\frac{d}{dx}\{f(x)g(x)\} = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$.

Do again the proof...

Example 0.0.18. What is the derivative of $(x^{2010} - x + 12)(x^3 + 32x)$?

Do: 5c/94; 12a/95 (or 13a/95).

2.3 Second derivatives, velocity, acceleration

Definition 0.0.9. The derivative of a derivative is called the second derivative; if f is given, f' (as you know) is called the derivative of f , and $(f)'$ is called the second derivative, and it is denoted by f'' .

Example 0.0.19. If $f(x) = x^{20110} - 76x^{65} + 12$, find f'' .

Do 3c/106.

Definition 0.0.10. If $s(t)$ denotes the distance an object moved over a period of time (t) , then $v(t) = s'(t)$ is the velocity (i.e., the rate of change of the displacement with respect to time (t)), and $a(t) = v'(t) = s''(t)$ is the acceleration (i.e., the rate of change of the velocity with respect to time (t)).

Do 3d/107, 12/109.

2.4 Chain Rule

Theorem 0.0.7. *Suppose that f and g are differentiable. One has that $\{f(g(x))\}' = f'(g(x))g'(x)$. In the other notation one has that if $y = f(u)$, and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$.*

Compute the derivatives of $f(x) = (x - 2010)^7$, $g(x) = \sqrt{30x + 31}$.

Power rule and chain rule gives

Theorem 0.0.8. *Power of a function rule $\{f(x)^n\}' = nf(x)^{n-1}f'(x)$.*

Do 9/118! and 7c/118! and 19/119!

Comments on lecture 5

2.5 Derivatives of quotients

The quotient rule is:

Theorem 0.0.9. *One has that $\left\{\frac{f(x)}{g(x)}\right\}' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$.*

Do: 9/125; 6d/125; 5d/125.

2.6 Rate of change problems

Typical problem:

The average ticket price for a hockey game at the Super Sens is 25 dollars. The average attendance is 4000. When the ticket price is raised to 27 dollars, attendance declines to an average of 3800 persons per performance. What should the ticket price be in order to maximize the revenue for the National Arts Centre? (Assume a linear demand curve.)

Solution:

We are done with chapter 2. Please read chapter 2 review, and then practice test (p.144—145).

A new Chapter... Chapter 3: CURVE Sketching

GOAL: with the help of the derivative(s) sketch the graph of a given function.

3.1 Increasing and Decreasing functions

Theorem 0.0.10. *If $f'(x) > 0$ on an interval then f is increasing on that interval.*

If $f'(x) < 0$ on an interval then f is decreasing on that interval.

Example 0.0.20. *If $f(x) = \frac{2x}{3x^2 + 12x}$ find the intervals where f is decreasing, and the intervals where f is increasing.*

Example 0.0.21. *Sketch a continuous function which satisfies the following conditions:*

$$f'(x) < 0 \text{ if } -1 < x < 3;$$

$$f'(x) > 0 \text{ if either } -1 > x \text{ or } x > 3;$$

$$f(0) = 0.$$

Comments on lecture 6

3.2 Maxima and Minima

In this section 3.2 we will focus on finding points where $f'(x) = 0$ (compare with 3.1). These points will give important information about maxima and minima of functions.

Definition 0.0.11. *A critical number of f is a number a in the domain of f such that either $f'(a) = 0$ or $f'(a)$ Does Not Exist. If a is a critical number, then the point $(a, f(a))$ on the graph of f is called a critical point.*

Example 0.0.22. Find the critical points of the function $g(x) = 2x^3 - 3x^2 - 12x + 5$.

Example 0.0.23. Do 3e/163.

Definition 0.0.12. The number a is called a local maximum value and $(a, f(a))$ is called a local maximum, if $f'(x)$ changes from POSITIVE to NEGATIVE as x increases from $x < a$ to $x > a$.

Definition 0.0.13. The number a is called a local minimum value and $(a, f(a))$ is called a local minimum, if $f'(x)$ changes from NEGATIVE to POSITIVE as x increases from $x < a$ to $x > a$.

Local maximum or minimum values are also called local extreme values or local extrema. Now let us look at the global definitions:

Definition 0.0.14. We say that a function f has an absolute maximum at a if $f(a) \geq f(x)$ for all x in the domain of f . The maximum value of f is $f(a)$.

Definition 0.0.15. We say that a function f has an absolute minimum at a if $f(a) \leq f(x)$ for all x in the domain of f . The minimum value of f is $f(a)$.

Revise:

Example 0.0.24. Find the local maximum points and local minimum points of the function $g(x) = 2x^3 - 3x^2 - 12x + 5$.

To get the absolute maximum and absolute minimum values of a given function $f : [a, b] \mapsto \mathbf{R}$ do the following:

- compute $f(a)$ and $f(b)$;
- compute the critical numbers, substitute the critical numbers you found in f ;
- the biggest value in the 2 steps above is the maximum value of f ; the smallest value in the 2 steps above is the minimum value of f .

One more time:

Example 0.0.25. Find the absolute maximum points and absolute minimum points of the function $g(x) = 2x^3 - 3x^2 - 12x + 5$ on $[-3, 3]$.

Do 11/164; or 2e/163.

3.3 Concavity and the second derivative test

Definition 0.0.16. — The graph of the function $f(x)$ is concave up on the interval $a < x < b$ if all the tangents to the graph of f on the interval are below the graph. The graph curves upward.

— The graph of the function $f(x)$ is concave down on the interval $a < x < b$ if all the tangents to the graph of f on the interval are above the graph. The graph curves downward.

— A point at which the graph changes from being concave up to concave down, or vice versa, is called an inflection point.

The link between concavity and the second derivative is given by the

Second derivative test:

Theorem 0.0.11 (Second derivative test). — A function f is concave up on an interval if $f''(x) > 0$ on that interval. If $f'(a) = 0$ and $f''(a) > 0$, then $(a, f(a))$ is a local minimum point.

— A function f is concave down on an interval if $f''(x) < 0$ on that interval. If $f'(a) = 0$ and $f''(a) < 0$, then $(a, f(a))$ is a local maximum point.

— If $f''(a) = 0$ and $f''(x)$ changes sign at $x = a$, then $(a, f(a))$ is an inflection point.

Try:

Example 0.0.26. Find the inflection points and the intervals of concavity of the function $g(x) = 2x^3 - 3x^2 - 12x + 5$.

Use the second derivative test for 7b/174.

Comments on lecture 7

3.4 Some simple rational functions

Recall that a rational fraction looks like this: $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are some polynomials, i.e., a quotient of two polynomials.

Definition 0.0.17. We say that the line $x = a$ is a vertical asymptote of $y = f(x)$ if either

$$\begin{aligned} \lim_{x \rightarrow a} f(x) &= \pm\infty \text{ or} \\ \lim_{x \rightarrow a^+} f(x) &= \pm\infty \text{ or} \\ \lim_{x \rightarrow a^-} f(x) &= \pm\infty. \end{aligned}$$

Let us draw some graphs for each situation described above. (Just Doo It!)

Then let us look at the following:

Example 0.0.27. Let $f(x) = \frac{1}{3x+12}$

For a rational function, a vertical asymptote may occur at the x -values for which the function is not defined. Our function $f(x)$ is not defined when the denominator $3x + 12 = 0$, in other words $x = -4$. Our claim is that: $x = -4$ is the vertical asymptote of $f(x)$. Indeed,

$$\lim_{x \rightarrow -4^+} f(x) = +\infty, \quad \lim_{x \rightarrow -4^-} f(x) = -\infty. \text{ Sketch yourself the graph...}$$

Definition 0.0.18. We say that the line $y = c$ is a horizontal asymptote of $y = f(x)$ (here c is a of course a number!) if either

$$\begin{aligned} \lim_{x \rightarrow -\infty} f(x) &= c \text{ or} \\ \lim_{x \rightarrow +\infty} f(x) &= c \end{aligned}$$

Note that polynomials do not have horizontal asymptotes! Look at $f(x) = x^3 - x^2 + 2010$. Compute $\lim_{x \rightarrow -\infty} f(x) = -\infty$ which is not a number, and $\lim_{x \rightarrow \infty} f(x) = \infty$ which is not a number.

Do: 3e/183 Consider $f(x) = \frac{x}{x^2-1}$. Find the vertical asymptotes, find the derivatives, sketch the graph. Are there any inflection points? Local maximums? Minimums?

If time, do the same for 1h/183.

3.5 Put all together: Curve Sketching

Sketch the curve of some polynomial functions and rational functions (see below) by using the first and the second derivatives, and the indications given in statements!

Example 0.0.28. Let $f(x) = x^3 - x^2 - 5x + 5$.

- Find the critical points of $f(x)$;
- Find the intervals where $f(x)$ is increasing, decreasing;
- Find the local maximum and local minimum points of $f(x)$;
- Find the intervals where $f(x)$ is concave up, concave down;
- Find the inflection points of $f(x)$;
- Does $f(x)$ have asymptotes? Why?
- Sketch the graph of $f(x)$.

Example 0.0.29. Let $f(x) = \frac{x}{x-5}$.

- Find the critical points of $f(x)$;
- Find the intervals where $f(x)$ is increasing, decreasing;
- Find the local maximum and local minimum points of $f(x)$;
- Find the intervals where $f(x)$ is concave up, concave down;
- Find the inflection points of $f(x)$;
- Does $f(x)$ have asymptotes? Why?
- Sketch the graph of $f(x)$.

Is it easy? Hard? More at the dgds, so please attend every FRI dgd!

Comments on lecture 8

3.6 Some optimization problems

Our goal is to learn how to solve applied optimization problems!

Question: How to solve applied optimization problems?

Answer(Plan):

- Introduce variables;
- Find conditions and functions;
- Use conditions to simplify function into one variable;
- Find the domain of the function;
- Find the critical points of the function;
- Find the (absolute) maximum (or minimum) point(s) of the function.

Example 0.0.30. A rectangular field is to be enclosed by a fence on 3 sides and by a straight stream on the fourth side. Find the dimensions of the field with maximal area that can be enclosed with 1000 feet of fencing.

Draw a picture ...

Introduce variables: say that l is the length of the rectangle, and that w is its width!

The condition in the statement says: $2w + l = 1000$. (Read again the statement!)

The question is: find the maximal value of the area A of the rectangle. But what is A ?

Recall that $A = lw$. Note that A depends on 2 variables: l and w . We want A to depend on only one variable. Note that $l = 1000 - 2w$. (see the condition one more time!)

Now, $A = lw = w(1000 - 2w)$, so in fact $A(w) = w(1000 - 2w)$.

The domain of our function is: $0 \leq w \leq 500$ because the width should be positive...

The critical points of your function are found as follows: $A'(w) = 0$, apply product rule and get $1 \times (1000 - 2w) + w \times (-2) = 0$, thus $1000 = 4w$, hence $w = \frac{1000}{4}$. We got that $w = 250$. So the critical number is 250, and the critical point is $(250, A(250)) = (250, 125000)$.

Review of what we did: $A : [0, 500] \rightarrow \mathbf{R}$, $A(250) = 125000$, $A(0) = 0$, $A(500) = 0$. The maximum value among the 3 we found is 125000.

Conclusion: If $w = 250$ (and thus $l = 1000 - 2w = 500$), we get that the maximal area of the field is 125000.

Do 2/201, 5, 4a/201.

Comments on lecture 9

Chapter 4.

4.1 (sine and cosine) and 4.2 (their derivatives)

Background on the sine and cosine functions:

— The sine of an angle is the ratio of the length of the opposite side to the length of the hypotenuse. The cosine of an angle is the ratio of the length of the adjacent side to the length of the hypotenuse. Draw yourself a picture!

— The sine and cosine functions give important examples of periodic functions, i.e., functions that repeat the values in regular intervals or periods. The period is 2π .

— Draw yourself the graphs!

— Identities involving the sine and cosine functions: $\sin^2(x) + \cos^2(x) = 1$, $\sin(x \pm h) = \sin(x)\cos(h) \pm \cos(x)\sin(h)$, $\cos(x + h) = \cos(x)\cos(h) - \sin(x)\sin(h)$, $\sin(2x) = 2\sin(x)\cos(x)$, $\cos(2x) = \cos^2(x) - \sin^2(x)$, $\sin(\frac{\pi}{2} - x) = \cos(x)$, $\cos(\frac{\pi}{2} - x) = \sin(x)$, etc.

Derivatives of the Sine and Cosine Functions:

Theorem 0.0.12. $\{\sin x\}' = \cos(x)$

Do the proof!

Theorem 0.0.13. $\{\cos x\}' = -\sin(x)$

Do the proof!

Example 0.0.31. Compute the derivatives of the functions (using what you learnt today, and all the other rules you already know, such as difference rule):

$(-2010 \sin(x))$, $\frac{3\pi}{200} \cos(x)$, $x^3 - 7 \cos(x) + 45 \sin(x)$.

Example 0.0.32. Find the equation of the tangent line to the graph of $f(x) = 2 \sin(x)$ at $x = \frac{\pi}{2}$.

Example 0.0.33. Find the points on the graph of $f(x) = -2 \cos(x)$ where the slope of the tangent line is $\sqrt{3}$.

Do from your book: 3f, 8, 4d,b/226.

4.3 Derivatives of sinusoidal functions

Chain rule and trig functions — see page 229

Power rule and trig functions — see same page

You may think of Product rule and trig functions, or Power rule and trig functions...

Do: page 231!

Comments on lecture 10

Chapter 5.

5.1 (Number e) and 5.2 (natural log)

Part I. Exponential Functions

The function $y = f(x) = a^x$, where $a > 0$ and $a \neq 1$, is called an exponential function with base a .

Note that exponential functions pass through the point $(0, 1)$, the domain is all real numbers (\mathbf{R}) and the range is all positive real numbers. There are two types of exponential functions depending on the base:

If the base $a > 1$, then $y = a^x$ is increasing. Draw the graph yourself;

If the base $a < 1$, then $y = a^x$ is decreasing. Draw the graph yourself.

Let us introduce the following number:

The number e is the unique real number such that the slope of the tangent line to the graph of the exponential function $y = f(x) = e^x$ at $x = 0$ is 1, in other words:

$$\lim_{h \rightarrow 0} \frac{e^{0+h} - e^0}{h} = 1.$$

An approximate value for e is 2.71828. There are many ways to evaluate e , for example:

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n.$$

Theorem 0.0.14. $\{e^x\}' = e^x$.

Do the proof!

Compute the derivatives of $f(x) = (x^{201} + x - \frac{1}{2})e^x$, $g(x) = e^{7x^2-2010}$, $h(x) = e^{-x} + 3e^x$,
 $k(x) = \frac{e^{-x}-e^x}{e^{-x}+e^x}$.

Part II. Logarithmic Functions

The inverse function (hope you were in class to learn about inverses!) of $y = a^x$, where $a > 0$ and $a \neq 1$, is called a logarithmic function with base a which is denoted by $y = \log_a x$. When $a = e$, we denote $\log_e x$ simply as $\ln x$ which is called the natural logarithmic function.

Using the definition of inverse functions we have $a^{\log_a x} = x$ and $\log_a(a^x) = x$.

LAWS: $\log_a(xy) = \log_a(x) + \log_a(y)$, $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$, $\log_a x = \frac{\ln(x)}{\ln(a)}$.

Note that if the base $a > 1$, then $y = \log_a x$ is increasing. Note that if the base $a < 1$, then $y = \log_a x$ is decreasing.

A simple note on the logarithmic functions is that they all pass the point $(1, 0)$, the domain is all positive real numbers and the range is all real numbers. This is due to the fact that $y = \log_a x$ is the inverse function of $y = a^x$. The graph of $y = \log_a x$ and the graph of $y = a^x$ are symmetric to each other about the line $y = x$.

Comments on lecture 11

We are still in Chapter 5!

5.3 and 5.4

We start by the following

Theorem 0.0.15. *If the functions $f(x)$ and $g(x)$ are **inverse** to each other, then one has that $g'(x) = \frac{1}{f'(g(x))}$.*

Indeed, from $f(g(x)) = x$ one gets by chain rule (yes, we take derivatives!) that $f'(g(x))g'(x) = 1$, so after dividing one obtains that $g'(x) = \frac{1}{f'(g(x))}$.

We are more interested in the following consequences

Theorem 0.0.16. $\{\ln(x)\}' = \frac{1}{x}$.

Indeed, just note that $\ln(x)$ is (by definition) the inverse of e^x . Thus by above Theorem, one obtains that $\{\ln(x)\}' = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$. (Recall that $(e^x)' = e^x$.)

Theorem 0.0.17. $\{\log_a(x)\}' = \frac{1}{\ln(a)x}$.

Indeed, $\log_a(x) = \frac{\ln(x)}{\ln(a)}$, and now take the derivative $\{\log_a(x)\}' = \left(\frac{\ln(x)}{\ln(a)}\right)' = \frac{1}{\ln(a)x}$ (what easy rule did we use?).

Hence we can get the following general

Theorem 0.0.18. *One has that $\{a^x\}' = a^x \ln(a)$.*

Indeed, since a^x is (by definition) the inverse of $\log_a(x)$, by today's first theorem (and the above theorem) one gets that $\{a^x\}' = \frac{1}{\frac{1}{\ln(a)a^x}} = a^x \ln(a)$.

OK, enough with the theory, let us play with some problems.

Exercise 0. Differentiate the functions $f(x) = 87^x$, $g(x) = 87^{7x-1}$, $k(x) = 87^{\sin(9x)}$.

Exercise 1. If $y(x) = 7^x(\sin(x) + 23x^4 - 12)$ find $y'(x)$.

Exercise 2. If $f(x) = 23^{2\cos(x) - \sin(2x)}$, find $f'(x)$.

Exercise 3. If $g(x) = \frac{20x^2 - 90x + 1}{e^{\cos(x)}}$, find $g'(x)$.

Exercise 4. Find the local maximum points and local minimum points (if any) of the function given by $f(x) = x^2e^x$. Find the intervals where $f(x)$ is concave up, concave down. Are there any inflection points? Any vertical or horizontal asymptotes? Please fill in the tables for x, f', f'', f .

If time do: 13/283; 3,2/282;

Next time: algebra

Comments on lecture 12

We are in Chapter 6!

6.1 and 6.2

Part I. Geometric Vectors

Chapter 6 gives a geometric perspective of vectors while chapter 7 gives an algebraic perspective of vectors.

Definition 0.0.19. *A scalar is a quantity that describes magnitude or size only. It does not include direction. Examples of scalars include number, temperature, area, distance, speed, mass.*

Definition 0.0.20. *A vector is a quantity that has both magnitude and direction. Examples of vectors include displacement, velocity, force.*

Geometrically we represent a vector as an arrow. The length of the arrow represents the magnitude of the vector and the direction of the arrow represents the direction of the vector. We will denote the vector in the following diagram as \overrightarrow{AB} , where A is the starting point (tail) and B is the end point (head). The magnitude of the vector \overrightarrow{AB} is denoted by $|\overrightarrow{AB}|$.

We can also use \vec{v} to denote a vector and use $|\vec{v}|$ to denote the magnitude of the vector.

We continue by giving the

Definition 0.0.21. *2 vectors are called **equivalent** if they have the same magnitude and the same direction.*

This definition tells us that the location of the vectors does not matter. Thus we can move the vectors around to meet our needs as long as we keep the magnitude and the direction of the vectors! Draw yourself a picture!

More

Definition 0.0.22. 2 vectors are called opposite vectors if they have the same magnitude but opposite direction.

Draw yourself a picture!

Definition 0.0.23. 2 vectors are called parallel if they have the same or opposite direction (but not necessarily the same magnitude).

Draw yourself a picture!

Part II. Vector Addition and Subtraction

If we are given two vectors \vec{a} and \vec{b} , we can define the vector addition $\vec{a} + \vec{b}$ which is a new vector obtained as follows:

Triangle Method:

— The tail of \vec{b} touches the head of \vec{a} (because of equivalent vectors, we can move the vector \vec{b} around);

— Draw from the tail of \vec{a} to the head of \vec{b} .

Please draw a picture!

Paralleloram Method:

— Connect the tails of \vec{a} and \vec{b} (because of equivalent vectors, we can move the vector \vec{b} around);

— Complete the parallelogram that has \vec{a} and \vec{b} as two of its sides;

— Draw the diagonal of the parallelogram.

Please draw a picture!

Vector subtraction is defined as follows:

Definition 0.0.24. Given 2 vectors \vec{a} and \vec{b} , we define $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$.

Part III. Properties of Vector Addition

Associative Property: $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

Commutative Property: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

Identity Property: If $\vec{0}$ is (by notation) the zero vector defined as having zero magnitude and no specific direction, then

$$\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$$

Exercise: simplify the following expressions: $(\vec{u} + \vec{v}) - \vec{u}$;

$$((\vec{p} + \vec{q}) - \vec{p}) - \vec{q}.$$

Do 4/325; 5/326; 12a/327; 9/326, 3/325.

Comments on lecture 13

We are still in Chapter 6!

6.3 + 6.4

Let us talk about

Scalar Multiplication

Given a vector \vec{v} and a scalar k , where k in \mathbf{R} , we can define the scalar multiplication $k\vec{v}$ which is a new vector obtained as follows:

— Magnitude or size : $|k\vec{v}| = |k||\vec{v}|$, the size of the new vector is $|k|$ times as that of \vec{v} ;

— Direction: If $k > 0$, then $k\vec{v}$ has the same direction as \vec{v} ;

— If $k < 0$, then $k\vec{v}$ has the opposite direction to \vec{v} ; So \vec{v} and $k\vec{v}$ are parallel.

DRAW some pictures!

PROPERTIES:

— Distributive Property: $k\{\vec{v} + \vec{u}\} = k\vec{v} + k\vec{u}$

— Associative Property: $(km)\vec{v} = k(m\vec{v})$

— Identity Property: $1\vec{v} = \vec{v}$

Theorem 0.0.19. (i) $(-1)\vec{v} = -\vec{v}$;

(ii) $(-k)\vec{v} = -k\vec{v}$;

(iii) $k(\vec{u} - \vec{v}) = k\vec{u} - k\vec{v}$.

Prove the theorem!

DO 2,8,19/334.

Definition 0.0.25. Given two vectors \vec{u} and \vec{v} and scalars k in \mathbf{R} and m in \mathbf{R} , then the vector $k\vec{u} + m\vec{v}$ is called a linear combination of \vec{u} and \vec{v} .

Example 0.0.34. $67\vec{u} - 23\vec{v}$ is a linear combination of \vec{u} and \vec{v} .

DO example 3/333.

6.4+6.5

Resolution of Vectors Into Rectangular Components

Definition 0.0.26. Given a vector \vec{v} , the rectangular components of \vec{v} are two vectors \vec{v}_1 and \vec{v}_2 satisfying (1) $\vec{v} = \vec{v}_1 + \vec{v}_2$ and (2) \vec{v}_1 and \vec{v}_2 are perpendicular to each other, i.e. the angle between \vec{v}_1 and \vec{v}_2 is $\pi/2$.

In this case, we say that \vec{v} resolves into \vec{v}_1 and \vec{v}_2 or $\vec{v}_1 + \vec{v}_2$ is a resolution of \vec{v} into rectangular components.

DRAW a picture!

Note that the vectors \vec{v}_1 and \vec{v}_2 ARE NOT UNIQUE!

DO: example 2/339 a)

Do: example 1/348

Do: example 4/341

Do: 12/350

Comments on lecture 14

Chapter 7 Cartesian Vectors

7.1 Cartesian vectors

What is a Cartesian vector?

What are unit vectors? Examples!

Vector Operations:

— Vector Addition

— Vector Subtraction

— Scalar Multiplication

Do some examples!

What is the magnitude (size) of a Cartesian vector? Examples!

Talk about cartesian vector between two points! Example!

7.2 Dot Product

Give the definition!

Work an example!

Give a theorem to help compute the dot product!

Try examples from book!

Comments on lecture 15

Chapter 7 Cartesian Vectors

7.2 Dot Product

If $\vec{u} = [31, 23]$ and $\vec{v} = [1, -2]$ find $\vec{u} \cdot \vec{v}$.

We have that the dot product is in this case: $31 \times 1 + 23 \times (-2) = -15$. So the angle between $\vec{u} = [31, 23]$ and $\vec{v} = [1, -2]$ is not acute, but obtuse.

Notation: $\vec{0} = [0, 0]$.

The properties of dot product are summarized in the following

Theorem 0.0.20. §1 $\vec{u} \cdot \vec{0} = 0$;

§2 $\vec{u} \cdot \vec{u} = |\vec{u}|^2$;

§3 $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$;

§4 $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$;

§5 $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$;

§5 $(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (k\vec{v})$.

Do 3/376; 8a/376; 10/376; 11/376; 12/376.

7.3 Some applications of Dot Product

Based on the previous section one has the following

Theorem 0.0.21. *The angle θ (in the interval $[0, \pi]$) between two nonzero vectors \vec{u} and \vec{v} is given by the formula*

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$$

What is the angle between $\vec{u} = [14, -2]$ and $\vec{v} = [1, 7]$?

Answer: $\pi/2$.

Recall from the previous lecture that 2 vectors are called orthogonal (or perpendicular) if the angle between them is $\pi/2$.

Theorem 0.0.22. *Two nonzero vectors are orthogonal if and only if their dot product is 0.*

Let us talk about **Vector Projections**.

Definition 0.0.27. *Given two vectors \vec{u} and \vec{v} . The projection of \vec{v} on \vec{u} , denoted by $\text{proj}_{\vec{u}} \vec{v}$, is the rectangular component of \vec{v} in the direction of \vec{u} .*

To compute such vector projections we need:

Definition 0.0.28. *The projection of \vec{v} on \vec{u} is given by $\text{proj}_{\vec{u}} \vec{v} = \frac{\vec{v} \cdot \vec{u}}{|\vec{u}|^2} \vec{u}$.*

Do 4,5/384.

Vectors in the three-space

A vector in the three-space can be represented algebraically by an ordered triple $\vec{u} = [a, b, c]$, where its tail is at the origin $O(0, 0, 0)$ and its head is at the point $P(a, b, c)$ on the 3 dimensional Cartesian system.

The usual operations are:

Vector Addition: $\vec{u} + \vec{v} = [a, b, c] + [m, n, p] = [a + m, b + n, c + p]$.

Vector Subtraction: $\vec{u} - \vec{v} = [a, b, c] - [m, n, p] = [a - m, b - n, c - p]$.

Magnitude (or size) of $\vec{u} = [a, b, c]$ is given by $|\vec{u}| = \sqrt{a^2 + b^2 + c^2}$.

Dot Product: $\vec{u} \cdot \vec{v} = [a, b, c] \cdot [m, n, p] = am + bn + cp$.

Do Example 4/393.

Do 9,16,29,12,7,8/399

Comments on lecture 16

7.5 Cross product + properties

We begin by

Definition 0.0.29. Given two vectors \vec{u} and \vec{v} in the 3 dimensional space, we define the **cross product** of \vec{u} and \vec{v} (in this order) to be the vector (in the 3 dimensional space): $(|\vec{u}||\vec{v}|\sin(\theta))\vec{n}$. We denote it by $\vec{u} \times \vec{v}$.

Here \vec{n} is the **unit vector** perpendicular to both \vec{u} and \vec{v} , and such that \vec{u} , \vec{v} and \vec{n} follow the right-hand rule.

Here θ is the angle between \vec{u} and \vec{v} , and moreover $0 \leq \theta \leq \pi$.

His size is given in the next

Theorem 0.0.23. $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin(\theta)$

Just think about the fact \vec{n} is a **unit** vector, and $\sin(\theta)$ is positive when $0 \leq \theta \leq \pi$.

After size was determined, from its definition we see that $\vec{u} \times \vec{v}$ is perpendicular to both \vec{u} and \vec{v} .

Geometric meaning: the size (magnitude) of $\vec{u} \times \vec{v}$ is the area of the parallelogram formed by \vec{u} and \vec{v} .

The Cross Product for Cartesian vectors is given by the following:

Theorem 0.0.24. If $\vec{u} = [u_1, u_2, u_3]$ and $\vec{v} = [v_1, v_2, v_3]$, then

$$\vec{u} \times \vec{v} = [u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1]$$

Example: compute $[1, 2, 3] \times [4, 5, 6]$.

Some properties of the cross product:

1. Assume that \vec{u} and \vec{v} are nonzero vectors. Then one has that $\vec{u} \times \vec{v} = \vec{0}$ IF AND ONLY IF \vec{u} and \vec{v} ARE COLLINEAR.

Compare with $\vec{u} \cdot \vec{v} = 0$ IF AND ONLY IF \vec{u} and \vec{v} ARE perpendicular!

2. One has that $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$. Thus the cross product is NOT commutative, BUT anti-commutative.

Compare with $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.

3. If k is a scalar, and \vec{u} , \vec{v} , \vec{w} are vectors then:

— distributive property: $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$ and $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$.

— associative property: $(k\vec{u}) \times \vec{v} = k(\vec{u} \times \vec{v}) = \vec{u} \times (k\vec{v})$.

Do: 7c, 9, 8, 11, 12, 4a/411

7.6 Applications of dot and cross product

Triple Scalar Product :

Definition 0.0.30. Given 3 vectors \vec{a} , \vec{b} , and \vec{c} in the 3 dimensional case we define the triple scalar product by $\vec{a} \cdot (\vec{b} \times \vec{c})$.

Its geometric meaning is: $|\vec{a} \cdot (\vec{b} \times \vec{c})|$ EQUALS the volume of the parallelepiped defined by the three vectors.

DO: 5, a,b/418; 2a/418; 4c/418.

Comments on lecture 17

Chapter 8 Lines + Planes

8.1

PART I. Equations of Lines in Two-Space

Even if lines and planes are geometrical objects, we can still use algebra for their analysis. Equations of lines and planes are their algebraic representations. The idea is very simple (and probably the only one):

any point on the lines or planes satisfies some equations.

Recall that in the well-known 2-space the slope intercept equation of a line is given by $y = mx + n$, where m is the slope of the line, and n is the y -intercept. You met this even in the previous chapters of Calculus...

The **scalar equation** is given by:

Definition 0.0.31. The scalar equation (also called the standard equation) of a line in 2-space is $ax + by + c = 0$; the vector $\vec{n} = [a, b]$ is called the normal vector to the line $ax + by + c = 0$. (Of course a , b and c are real numbers.)

Definition 0.0.32. A normal vector to a line is a vector that is perpendicular to that line.

Example 0.0.35. Check that the vector $\vec{n} = [1, 1]$ is normal to the line $x + y + 1 = 0$. Of course you could use the dot product from the previous chapter!

The **vector equation** is given by:

Definition 0.0.33. The vector equation of a line in 2-space is given by

$\vec{r} = \vec{r}_0 + t\vec{m}$ or $[x, y] = [x_0, y_0] + t[m_1, m_2]$ where:

t is a scalar (number);

$\vec{r} = [x, y]$ is a vector corresponding to any unknown point on the line;

$\vec{r}_0 = [x_0, y_0]$ is a vector corresponding to any known point on the line;

$\vec{m} = [m_1, m_2]$ is a direction vector parallel to the line.

Example 0.0.36. Find the vector equation of the line $x + 2y + 1 = 0$.

First find a point on the line. Is $P(-1, 0)$ good? Then choose $\vec{r}_0 = [x_0, y_0] = [-1, 0]$. Are you able to get \vec{m} ? Do it!

The **Parametric Equation** is given by:

Definition 0.0.34. The parametric equation of a line in 2-space is given by $x = x_0 + tm_1$, $y = y_0 + tm_2$ where t is the parameter.

Example 0.0.37. Look at the line $x = 3 + 2t$, $y = -5 + 4t$. Find two points on the line. Write the vector equation of the line. Write the scalar equation of the line.

PART II. *Equations of Lines in Three-Space*

Generalizing our above talk we give the following definition for The Vector Equation of a line in the 3-space:

Definition 0.0.35. *The vector equation of a line in the 3-space is given by:*

$$\vec{r} = \vec{r}_0 + t\vec{m} \text{ or } [x, y, z] = [x_0, y_0, z_0] + t[m_1, m_2, m_3] \text{ where:}$$

t is a scalar (number);

$\vec{r} = [x, y, z]$ is a vector corresponding to any unknown point on the line;

$\vec{r}_0 = [x_0, y_0, z_0]$ is a vector corresponding to any known point on the line;

$\vec{m} = [m_1, m_2, m_3]$ is a direction vector parallel to the line.

and for The Parametric Equation of a line in the 3-space:

Definition 0.0.36. *The parametric equation of a line in 3-space is given by: $x = x_0 + tm_1$, $y = y_0 + tm_2$, $z = z_0 + tm_3$ where t is the parameter.*

Example 0.0.38. *A line passes through points $(2, 1, 0)$ and $(1, 0, 3)$.*

- find the vector equation of the line;
- find the parametric equation of the line;
- is the point of coordinates $(3, 1, 2)$ on the line?

Do problems from pages 437 and 438! More at the dgds, so please attend it!

Comments on lecture 18

Chapter 8 Lines + Planes

8.2 Equations of the planes

Definition 0.0.37. *The scalar equation (also called the standard equation) of a plane in (of course) three-dimensional space is $ax + by + cz + d = 0$, where $\vec{n} = [a, b, c]$ is a normal vector to the plane.*

where by definition:

Definition 0.0.38. *A normal vector to a plane is a vector that is perpendicular(orthogonal) to that plane.*

Look at the example:

Example 0.0.39. *Consider the plane $x + y + z + 1 = 0$. Show yourself that $\vec{n} = [1, 1, 1]$ is a normal vector to this plane.*

HOW? Pick 2 distinct points on that plane, say $P(-1, 0, 0)$ and $Q(0, -1, 0)$, compute then \vec{PQ} , then compute the dot product:

$$\vec{PQ} \cdot [1, 1, 1]. \text{ Do you get 0?}$$

Vector Equation is given by the following:

Definition 0.0.39. *The vector equation of a plane in three-space is given by:*

$$\vec{r} = \vec{r}_0 + t\vec{a} + s\vec{b} \text{ or by}$$

$$[x, y, z] = [x_0, y_0, z_0] + t[a_1, a_2, a_3] + s[b_1, b_2, b_3], \text{ where}$$

t, s 2 are scalars,

$\vec{r} = [x, y, z]$ is a vector corresponding to any unknown point on the line,

$\vec{r}_0 = [x_0, y_0, z_0]$ is a vector corresponding to any known point on the line,

$\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$ are two non collinear direction vectors parallel to the plane.

The Parametric Equation are given by:

Definition 0.0.40. The parametric equation of a plane in three-space is:

$$x = x_0 + ta_1 + sb_1,$$

$$y = y_0 + ta_2 + sb_2,$$

$$z = z_0 + ta_3 + sb_3, \text{ where } t, s \text{ are parameters (i.e., real numbers).}$$

Example 0.0.40. Consider the plane with direction vectors $\vec{a} = [1, 2, -3]$ and $\vec{b} = [2, -1, 4]$ passing through the point $P(2, 0, 1)$.

- (a) Find the vector equation of the plane;
- (b) Compute the parameter equation of the plane,
- (c) Find another point on the plane,
- (d) Find the y -intercept of the plane.

DO: 10c on 451; 5a; 6b;

From 8.3 do:

2a; 3 on 459

then do: 6b on 459 7d on 459

and:

10a, 11, 12d on 460

Comments on lecture 19

Chapter 8 Lines + Planes

8.3 Properties of planes

see the above exercises: from 8.3 do:

2a; 3 on 459

then do: 6b on 459 7d on 459

and:

10a, 11, 12d on 460.

8.4 Intersection of Lines in Two-Space and 3-space

Part I. Intersection of Two Lines in Two-Space

From the point of view of geometry there are three possibilities for the intersection of two lines in two-space:

- Intersect at a point
- Coincident
- Parallel

From the point of view of algebra we must find the solutions of the following system of equations:

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

EXAMPLES of each case:

$$(i) \ x - y + 1 = 0$$

$$2x + y + 4 = 0$$

$$(ii) \ x - y + 1 = 0$$

$$2x - 2y + 2 = 0$$

$$(iii) \ x - y + 1 = 0$$

$$x - y - 201 = 0$$

Part II. Intersection of Two Lines in Three-Space

From the point of view of geometry there are three possibilities for the intersection of two lines in 3-space:

- Intersect at a point
- Coincident
- Parallel
- Skew (not parallel and do not intersect)

From the point of view of algebra we must find the solutions of the following system of equations:

$$[x, y, z] = [x_0, y_0, z_0] + s[m_1, m_2, m_3]$$

$$[x, y, z] = [a_0, b_0, c_0] + t[n_1, n_2, n_3].$$

EXAMPLES:

$$(i) \ [x, y, z] = [7, 2, -6] + s[2, 1, -3]$$

$$[x, y, z] = [3, 9, 13] + t[1, 5, 5]$$

There is only one point where they intersect! Do it!

Part III. 8.5 Intersection of a Line and a Plane in Three-Space

From the point of view of geometry there are three possibilities for the intersection of a Line and a Plane in Three-Space:

- Intersect at a point
- The line lies on the plane
- Parallel

From the point of view of algebra we must find the solutions of a system of equations

$$[x, y, z] = [a_0, b_0, c_0] + t[n_1, n_2, n_3],$$

$$ax + by + cz + d = 0$$

and we can get only these cases:

- Unique solution: intersect at a point
- Infinitely many solutions: the line lies on the plane
- No solution at all: parallel

EXAMPLE: Find the intersection of the following objects:

$$[x, y, z] = [5, -5, 2] + t[2, -5, 3] \text{ and } 9x + 13y - 2z = 29.$$

4,5,6 on 471

3,4,5 on 480

Comments on lecture 20

Chapter 8 Lines + Planes

8.6 Intersections of planes

Part I. Intersection of Two Planes in Three-Space

From the point of view of geometry there are three possibilities for the intersection of two planes in three-space:

- Intersect in a line
- Coincident
- Parallel

From the point of view of algebra there are two possibilities for the solutions of the following system of equations:

$$a_1x + b_1y + c_1z + d_1 = 0$$

$$a_2x + b_2y + c_2z + d_2 = 0$$

— Infinitely many solutions (intersect in a line or coincident)

— No solution (parallel)

Example 0.0.41. Find the intersection of the planes:

$$2x - y + z - 1 = 0$$

$$x + y + z - 6 = 0$$

Note that the 2 normal vectors are not parallel, THUS the planes are not parallel. Note that they are not coincident!

To find the intersection line: 1) from the first 2 equations we get $x - 2y + 5 = 0$; 2) Say $y = t$, where t is a parameter; 3) find x from 1) as follows: $x = 2t - 5$; 4) the second plane gives us $z = 11 - 3t$.

5) From this parametric equations find the vector equation!!!

Part II. Intersection of 3 Planes in Three-Space

From the point of view of geometry there are 8 possibilities for the intersection of 3 planes in three-space:

- 1) Intersect at a point
- 2) Intersect in a line
- 3) Three planes are coincident
- 4) Two planes are coincident and the third plane is not parallel
- 5) Three planes are parallel
- 6) Two planes are coincident and the third plane is parallel to the first two planes
- 7) Two planes are parallel and the third plane is not parallel
- 8) Pairs of planes intersect in lines that are parallel

From the point of view of algebra there are three possibilities for the solutions of the following system of equations:

$$a_1x + b_1y + c_1z + d_1 = 0$$

$$a_2x + b_2y + c_2z + d_2 = 0$$

$$a_3x + b_3y + c_3z + d_3 = 0$$

— Unique solution 1

— Infinitely many solutions 2-4

— No solution 5-8

Example 0.0.42. The following three planes intersect in a line. Determine the parametric equation and the vector equation of the line.

$$x - 5y + 2z - 10 = 0$$

$$x + 7y - 2z + 6 = 0$$

$$8x + 5y + z - 20 = 0$$

AS above: from the first 2 equations one gets $12y - 4z + 16 = 0$. Say $z = t$, where t is a parameter. Then $y = \frac{1}{3}t - \frac{4}{3}$. USING: — The following three planes intersect in a line — plug in the first equation and get x . Then write the vector equation.

Do 2a,b and 3a, 4b on 491.