MAT 1339 C **Assignment 4** (Due Dec. 1st, 5:30) **Student Number**:

Name:

Problem 1: Consider the line $[x, y, z] = [1, 2, 3] + t[5, 6, 7]$.

(a) Is the point $R(3, 2, 1)$ on this line?

(b) Write the parametric equation of this line.

(c) Does the line $[x, y, z] = [-1, 2, -3] + t[0, 1, 2]$ intersect the line $[x, y, z] = [1, 2, 3] + s[5, 6, 7]$?

(d) Give a vector $[a, b, c]$ in the 3 space that is parallel to the line $[x, y, z] = [1, 2, 3] + t[5, 6, 7]$ and such that $a < 0$, $b < 0$ and $c < 0$.

Work:

(a) No, because there is no t satisfying $[3, 2, 1] = [1 + 5t, 2 + 6t, 3 + 7t]$.

(b) $\begin{cases} x = 1 + 5t \\ y = 2 + 6t \\ z = 3 + 7t \end{cases}$

(c) No, because there are no s and t satisfying

$$\begin{cases} -1 = 1 + 5s \\ 2 + t = 2 + 6s \\ -3 + 2t = 3 + 7s \end{cases}$$

In fact, these are skew lines.

(d) As $[5, 6, 7]$ is a direction vector of the line, $t[5, 6, 7]$ for any $t < 0$ gives the required vector. For instance, $[-5, -6, -7]$ is a vector parallel to the line such that all components are negative.

Problem 2: Let $\overrightarrow{u} = [2, -1, 2]$ and $\overrightarrow{v} = [\frac{\sqrt{7}}{2\sqrt{5}}, \frac{\sqrt{7}}{\sqrt{5}}, \frac{3}{2}]$ be two vectors in three dimensional space.

(i) Find the angle between $\overrightarrow{u}$ and $\overrightarrow{v}$.

(ii) Find two unit vectors that are orthogonal to both $\overrightarrow{u}$ and $\overrightarrow{v}$.

Work:

(i) Let θ be the angle between $\overrightarrow{u}$ and $\overrightarrow{v}$. By definition of dot product, we have

$$\overrightarrow{u} \cdot \overrightarrow{v} = |\overrightarrow{u}||\overrightarrow{v}| \cos\theta.$$

Since $|\overrightarrow{u}| = 3$, $|\overrightarrow{v}| = 2$, and $\overrightarrow{u} \cdot \overrightarrow{v} = 3$, we get $\cos\theta = 1/2$, hence $\theta = \pi/3$.

(ii) As the cross product $\overrightarrow{u} \times \overrightarrow{v}$ is the orthogonal to both $\overrightarrow{u}$ and $\overrightarrow{v}$, the two unit vectors are

$$\pm\frac{\overrightarrow{u} \times \overrightarrow{v}}{|\overrightarrow{u} \times \overrightarrow{v}|} = \frac{\pm 1}{\sqrt{(-\frac{3}{2} - \frac{2\sqrt{7}}{\sqrt{5}})^2 + (\frac{\sqrt{7}}{\sqrt{5}} - 3)^2 + (\frac{\sqrt{35}}{2})^2}}[-\frac{3}{2} - \frac{2\sqrt{7}}{\sqrt{5}}, \frac{\sqrt{7}}{\sqrt{5}} - 3, \frac{\sqrt{35}}{2}]$$

Problem 3: Suppose that the volume of the parallelepiped defined by $\overrightarrow{u} = [1,2,3]$, $\overrightarrow{v} = [2,3,4]$, and $\overrightarrow{w} = [5,6,x]$, is 1. Find all x.

Work:

Note that the volume of the parallelepiped is $|(\overrightarrow{u} \times \overrightarrow{v}) \cdot \overrightarrow{w}|$. As $\overrightarrow{u} \times \overrightarrow{v} = [-1,2,-1]$, we have $1 = |(\overrightarrow{u} \times \overrightarrow{v}) \cdot \overrightarrow{w}| = |[-1,2,-1] \cdot [5,6,x]| = |-5+12-x|$. Therefore, $x = 6$ or 8.

Problem 4: Find the projection of $\overrightarrow{u} = [1,2,4]$ on $\overrightarrow{v} = [3,1,2]$.

Work:

$$\text{Proj}_{\overrightarrow{v}} \overrightarrow{u} = (\frac{\overrightarrow{u} \cdot \overrightarrow{v}}{\overrightarrow{v} \cdot \overrightarrow{v}})\overrightarrow{v} = (\frac{[1,2,4] \cdot [3,1,2]}{[3,1,2] \cdot [3,1,2]})[3,1,2] = \frac{13}{14}[3,1,2].$$

Problem 5: Find the distance from the point $P = (1, 1, 6)$ to the line

$$x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

Work: (Solution 1) Pick a point $S = (1, 3, 0)$ on the line. Let $\overrightarrow{v} = [1, -1, 2]$ be a direction vector of the line. Then, the distance d is given by

$$d = \frac{|\overrightarrow{SP} \times \overrightarrow{v}|}{|\overrightarrow{v}|}.$$

As $\overrightarrow{SP} = \overrightarrow{OP} - \overrightarrow{OS} = [1-1, 1-3, 6-0] = [0, -2, 6]$ and $\overrightarrow{SP} \times \overrightarrow{v} = [2, 6, 2]$, we get

$$d = \frac{\sqrt{44}}{\sqrt{6}} = \sqrt{\frac{22}{3}}.$$

(Solution 2) Let $S = (1+t, 3-t, 2t)$ be a point on the line. Suppose that $\overline{PS}$ is the distance between the point P and the line. Then $\overrightarrow{PS}$ is orthogonal to the direction vector $\overrightarrow{v} = [1, -1, 2]$ of the line. Therefore, we have

$$\overrightarrow{PS} \cdot [1, -1, 2] = 0. \tag{1}$$

As $\overrightarrow{PS} = (1 + t - 1, 3 - t - 1, 2t - 6)$, we get $t = \frac{7}{3}$ by (1). Hence, we have the distance,

$$|\overrightarrow{PS}| = \sqrt{(1 + \frac{7}{3} - 1)^2 + (3 - \frac{7}{3} - 1)^2 + (2 \cdot \frac{7}{3} - 6)^2} = \sqrt{\frac{22}{3}}.$$