MAT 1339 A **Assignment 1** (Due THU. Sept. 30th, 11:30) **Student Number**:

Name:

Problem 1: [1 mark] Using the definition of a derivative find $f'(x)$ if $f(x) = x^3 - 2x + 2010$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^3 - 2(x+h) + 2010 - (x^3 - 2x + 2010)}{h} \\ &= \lim_{h \to 0} \frac{(3xh + 3x^2 + h^2 - 2)h}{h} \\ &= \lim_{h \to 0} 3xh + 3x^2 + h^2 - 2 \\ &= 3x^2 - 2. \end{aligned}$$

Problem 2: [1 mark] Using the rules of differentiation find the derivative of $g(x) = 2x^{2010} - \frac{1}{2}x^{2000} + \frac{10}{x^6}$.

Solution:

(We are using POWER RULE!)

$$\begin{aligned} f'(x) &= (2x^{2010})' - (\frac{1}{2}x^{2000})' + (\frac{10}{x^6})' \\ &= 2 \cdot 2010 \cdot x^{2010-1} - \frac{1}{2} \cdot 2000 \cdot x^{2000-1} + 10 \cdot (-6) \cdot x^{-6-1} \\ &= 4020x^{2009} - 1000x^{1999} - \frac{60}{x^7}. \end{aligned}$$

Problem 3: [1 mark] If $g(x) = 2x^6 - 12x^3$ and $f(x) = 12x^3 + 4x^4$ find the derivative of $\frac{f(x)}{g(x)}$.

Solution:

(We are using QUOTIENT RULE!)

$$\begin{aligned}
\Big(\frac{f(x)}{g(x)}\Big)' &= \Big(\frac{12x^3 + 4x^4}{2x^6 - 12x^3}\Big)' \\
&= \Big(\frac{(12 + 4x)x^3}{(2x^3 - 12)x^3}\Big)' \\
&= \Big(\frac{12 + 4x}{2x^3 - 12}\Big)' \\
&= \frac{(12 + 4x)'(2x^3 - 12) - (12 + 4x)(2x^3 - 12)'}{(2x^3 - 12)^2} \\
&= \frac{4(2x^3 - 12) - (12 + 4x) \cdot 6x^2}{(2x^3 - 12)^2} \\
&= \frac{-4x^3 - 18x^2 - 12}{(x^3 - 6)^2}.
\end{aligned}$$

Problem 4: [1 mark] If $f(x) = 2x - x^{2010}$ and $g(x) = x^{23} - 2010 + x^{22}$ find the derivative of $f(x)g(x)$.

Solution:

(We are using PRODUCT RULE!)

$$\begin{aligned}
(f(x)g(x))' &= \big((2x - x^{2010})(x^{23} - 2010 + x^{22})\big)' \\
&= (2x - x^{2010})'(x^{23} - 2010 + x^{22}) + (2x - x^{2010})(x^{23} - 2010 + x^{22})' \\
&= (2 - 2010x^{2009})(x^{23} - 2010 + x^{22}) + (2x - x^{2010})(23x^{22} + 22x^{21}).
\end{aligned}$$

Problem 5: [2 marks] If $f(x) = \begin{cases} x + 2010, & \text{if } x < 3 \\ -2013, & \text{if } x = 3 \\ x^2 + 2004, & \text{if } x > 3 \end{cases}$

(a)[1 mark] find $\lim_{x\to 3^-} f(x)$, $\lim_{x\to 3^+} f(x)$, $\lim_{x\to 3} f(x)$;

(b)[1 mark] is f continuous at 3?

Solution: (a)

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x + 2010 = 2013,$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x^2 + 2004 = 2013,$$

$$\text{As } \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x), \lim_{x\to 3} f(x) = 2013.$$

(b)

$$\text{As } 2013 = \lim_{x\to 3} f(x) \neq f(3) = -2013,\ f(x) \text{ is discontinuous at } x = 3.$$

Problem 6: [2 marks] Suppose that $a_1 = 1$ and that $a_n = \frac{a_{n-1}}{n}$ for $n \geq 2$.

(i)[0.5 marks] Find a_5.

$$\boxed{a_5 = \frac{a_4}{5} = \frac{a_3}{4\cdot 5} = \frac{a_2}{3\cdot 4\cdot 5} = \frac{1}{2\cdot 3\cdot 4\cdot 5}.}$$

(ii)[0.5 marks] What statement (circle only one) is true about the sequence $\{a_n\}_{n=1}^{\infty}$?

(a) $\{a_n\}_{n=1}^{\infty}$ is increasing

(b) $\{a_n\}_{n=1}^{\infty}$ is decreasing

(c) $\{a_n\}_{n=1}^{\infty}$ is neither decreasing nor increasing

(iii)[0.5 marks] Why does the limit: $\lim_{n\to\infty} a_n$ exist? Give a 1 line answer!

Solution:

The sequence a_n becomes arbitrarily close to 0 for large values of n.

OR as a_n is bounded below by 0 and decreasing, it is has a limit.

OR for any $\epsilon > 0$, there exists an integer $n_0 > \frac{1}{\epsilon}$ such that for any $n > n_0$ we have $a_n = \frac{1}{1\times 2\times\cdots\times n} < \frac{1}{n} < \epsilon$.

(In other words: any integer n_0 that is bigger than $\frac{1}{\epsilon}$ will do the trick!)

(iv)[0.5 marks] What is $\lim_{n\to\infty} a_n$?

$$\boxed{\lim_{n\to\infty} a_n = 0}$$

Problem 7: [2 marks] Find the equation of the tangent line to the graph of $f(x) = \frac{x}{x+2}$ at the point $(1, \frac{1}{3})$. Hint: Recall that such an equation has the form $y = mx + n$. What is the meaning of m? Find m and n.

Solution:

As $f'(x) = \dfrac{x'(x+2) - x(x+2)'}{(x+2)^2} = \dfrac{x+2-x}{(x+2)^2} = \dfrac{2}{(x+2)^2}$, the slope of tangent line at $x = 1$ is $f'(1) = \dfrac{2}{9}$.

Since the tangent line having slope $\dfrac{2}{9}$ passes through $(1, \dfrac{1}{3})$, the equation is $y - \dfrac{1}{3} = \dfrac{2}{9}(x-1)$(or $y = \dfrac{2}{9}x + \dfrac{1}{9}$).