

**Calcul différentiel et intégral pour les sciences de la vie I**  
**MAT1730**  
**Test 1**

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**Question 1**

4 points Find the derivative of the following function :  $f(x) = \frac{x^5 \ln(x)}{4x + 6e^x}$  .

**Solution:**

We have  $f(x) = g(x)/h(x)$ , where  $g(x) = x^5 \ln(x)$  and  $h(x) = 4x + 6e^x$ . Since

$$g'(x) = 5x^4 \ln(x) + x^5 \left(\frac{1}{x}\right) = 5x^4 \ln(x) + x^4 = x^4(5 \ln(x) + 1) ,$$

we have

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{h^2(x)} = \frac{x^4(5 \ln(x) + 1)(4x + 6e^x) - x^5 \ln(x)(4 + 6e^x)}{(4x + 6e^x)^2}$$

**Question 2**

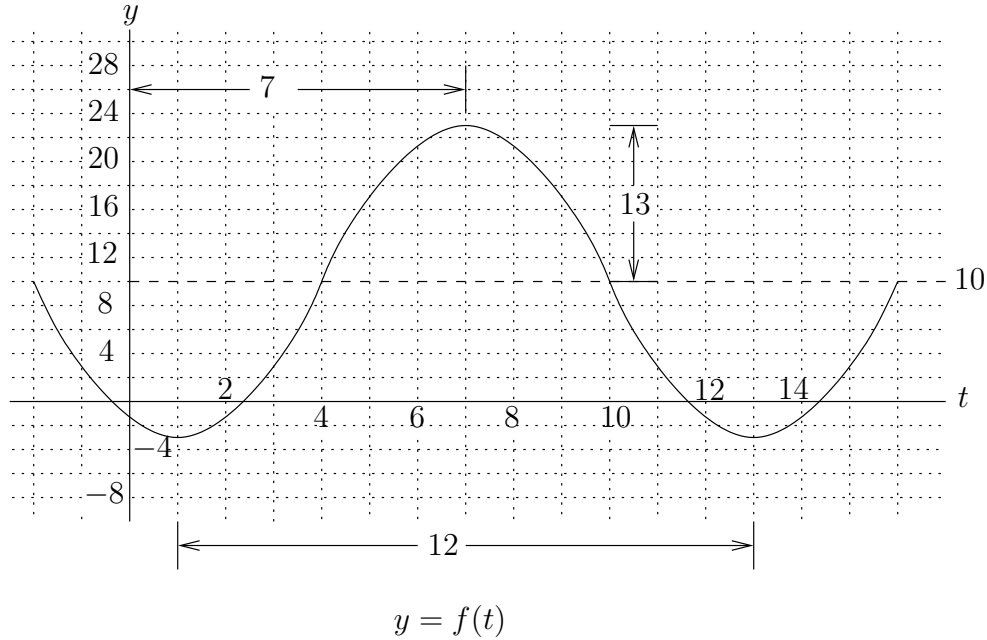
4 points Over the course of a year, the city of Ottawa has its highest average monthly temperature of  $23^\circ\text{C}$  in August and its lowest monthly average of  $-3^\circ\text{C}$  in February. Assume that temperature varies sinusoidally over a period of one year. Find the parameters in the standard cosine description , i.e.,

$$f(x) = M + A \cos\left(\frac{2\pi}{P}(t - T)\right) ,$$

where  $t$  is in months, and  $t = 0$  correspond to the month of January. Draw the graph of the function and identify the four parameters  $A, B, \Phi, T$  in the graph. Give the names of the four parameters.

**Solution:**

The mean is  $M = \frac{23 - 3}{2} = 10$ , the amplitude is  $A = \frac{23 + 3}{2} = 13$ , the period is 12 months and the phase is  $T = \frac{7}{12}$  months. We get the following graph.



### Question 3

4 points a) Find the following limit without using a calculator

$$\lim_{x \rightarrow -2} \frac{|x + 3| - 1}{x^2 - 4}.$$

b) Does the following limit exist? If yes, give the limit, if not, justify your answer. If you need a calculator to work out the answer, give at least 4 values of  $x$  that you tried.

$$\lim_{x \rightarrow 2} \frac{|x + 3| - 1}{x^2 - 4}.$$

### Solution:

For  $x > -3$ ,  $|x + 3| = x + 3$  and we have

$$\frac{|x + 3| - 1}{x^2 - 4} = \frac{x + 3 - 1}{x^2 - 4} = \frac{x + 2}{(x - 2)(x + 2)} = \frac{1}{x - 2}.$$

Since  $f(x) = 1/(x - 2)$  is continuous at  $x = -2$ , we have

$$\lim_{x \rightarrow -2} \frac{|x + 3| - 1}{x^2 - 4} = \lim_{x \rightarrow -2} \frac{1}{x - 2} = \frac{1}{-2 - 2} = -\frac{1}{4}.$$

b) As we saw in (a), for  $x > -3$ ,  $|x + 3| = x + 3$  and we have

$$\frac{|x + 3| - 1}{x^2 - 4} = \frac{x + 3 - 1}{x^2 - 4} = \frac{x + 2}{(x - 2)(x + 2)} = \frac{1}{x - 2}.$$

However,

$n$	$2 - \frac{1}{n}$	$\frac{ x+3 -1}{x^2-4} = \frac{1}{x-2}$
1	1	-1
2	1.5	-2
3	5/3	-3
$\vdots$	$\vdots$	$\vdots$
$n$	$2 - \frac{1}{n}$	$-n$
$\downarrow$	$\downarrow$	$\downarrow$
$\infty$	2	$-\infty$

and

$n$	$2 + \frac{1}{n}$	$\frac{ x+3 -1}{x^2-4} = \frac{1}{x-2}$
1	3	1
2	2.5	2
3	7/3	3
$\vdots$	$\vdots$	$\vdots$
$n$	$2 + \frac{1}{n}$	$n$
$\downarrow$	$\downarrow$	$\downarrow$
$\infty$	2	$\infty$

We conjecture that  $\lim_{x \rightarrow 2^+} \frac{|x+3|-1}{x^2-4} = \infty$  and  $\lim_{x \rightarrow 2^-} \frac{|x+3|-1}{x^2-4} = -\infty$ . Since

$$\lim_{x \rightarrow 2^+} \frac{|x+3|-1}{x^2-4} \neq \lim_{x \rightarrow 2^-} \frac{|x+3|-1}{x^2-4},$$

the limit doesn't exist.

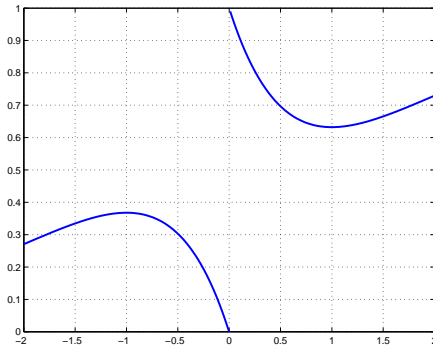
#### Question 4

4 points Use the definition of the derivative to calculate the derivative of the function  $f(x) = (18x - 11)^2 + 2009$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{((18(x+h) - 11)^2 + 2009) - ((18x - 11)^2 + 2009)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(18x + 18h - 11)^2 - (18x - 11)^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{(18x - 11)^2 + 36h(18x - 11) + (18h)^2 - (18x - 11)^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{36h(18x - 11) + (18h)^2}{h} = \lim_{h \rightarrow 0} (36(18x - 11) + 18^2h) = 36(18x - 11) \end{aligned}$$

#### Question 5

4 points Consider the graph of a function  $f$  below.



Find the critical points of  $f$ . Determine the intervals where  $f'$  is positive and the intervals where  $f'$  is negative.

**Solution:**

We have three critical points : at  $x = -1$  and  $x = 1$ , where the derivative is null, and at  $x = 0$ , where the derivative doesn't exist.

The function is decreasing for  $-2 \leq x < -1$  and  $0 < x < 1$ , and the functions is increasing for  $-1 < x < 0$  and  $1 < x \leq 2$ .

**Question 6**

8 points The human metabolism will break down morphine in the blood stream to 60% of its original amount within a 4-hour period. A patient is given a dose of 8 mg of morphine every 4 hours. The DTDS for the amount of morphine in the patient's body is given by

$$M_{n+1} = 0.6M_t + d = 0.6M_n + 8 \quad , \quad n = 0, 1, 2, \dots$$

where  $t$  is measured in 4-hour intervals and morphine is measured just after a new dose has been administered.

- What is the updating function of the DTDS?
- Find the fixed point of the DTDS if there is one.
- Find the general solution formula for the DTDS, i.e.,  $M_t = \dots$
- Graph the updating function and draw the cobwebbing, starting from  $M_0 = 5$  for at least 4 steps.
- Is the fixed point stable or unstable?
- The doctor wants to adjust the dose  $d$  so that the patient has 30 mg of morphine in the blood stream in the long run. What should the dose  $d$  be in that case?

**Solution:**

a) The generatrice function is  $f(x) = 0.6x + 8$ .

b)  $p$  is an steady state if  $f(p) = p$ . Thus,

$$f(p) = p \Rightarrow p = 0.6p + 8 \Rightarrow 0.4p = 8 \Rightarrow p = 20 .$$

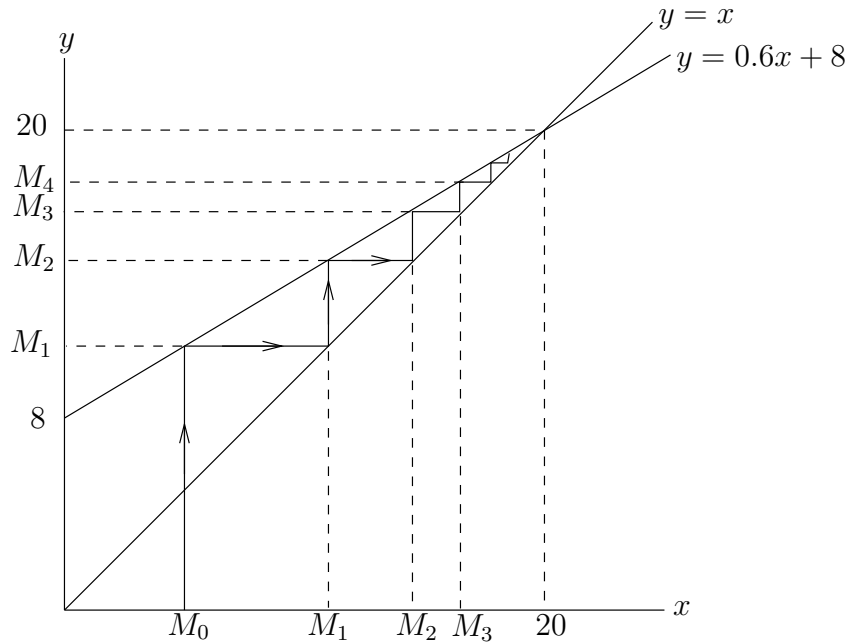
c) The general solution is

$$M_n = 0.6^n(M_0 - p) + p = 0.6^n(M_0 - 20) + 20 \quad , \quad n = 0, 1, 2, \dots$$

Another form for the solution is

$$M_n = 0.6^n M_0 + 8(1 + 0.6 + 0.6^2 + 0.6^3 + \dots + 0.6^{n-1}) \quad , \quad n = 1, 2, \dots$$

d)



e) Since we have a linear system of the form  $M_{n+1} = rM_n + b$  with  $|r| < 1$ , the equilibrium is stable. The stability is also obvious from cobwebbing as done in (d).

f) If we want the patient to have 30 mg of morphine in the blood in the long run, we must have an equilibrium point  $p$  such that  $0.6p > 30$ ; namely,  $p > 50$ . We have  $M_{n+1} = 0.6M_n + b$ . The equilibrium point  $p$  is given by  $p = 0.6p + b$ . Thus,  $p = 5b/2$ . We need to choose  $p = 5b/2 > 50$ ; namely,  $b > 20$  mg per four hours.

We need to choose a dose  $b$  larger than 20 mg per four hours.