Calcul différentiel et intégral pour les sciences de la vie I
MAT1730
Test 1

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Question 1

4 points Find the derivative of the following function : $f(x) = \dfrac{x^3 \ln(x)}{3x + 7e^x}$.

Solution:
We have $f(x) = g(x)/h(x)$, where $g(x) = x^3 \ln(x)$ and $h(x) = 3x + 7e^x$. Since

$$g'(x) = 3x^2 \ln(x) + x^3 \left(\frac{1}{x}\right) = 3x^2 \ln(x) + x^2 = x^2(3\ln(x) + 1) ,$$

we have

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{h^2(x)} = \frac{x^2(3\ln(x) + 1)(3x + 7e^x) - x^3 \ln(x)(3 + 7e^x)}{(3x + 7e^x)^2}$$

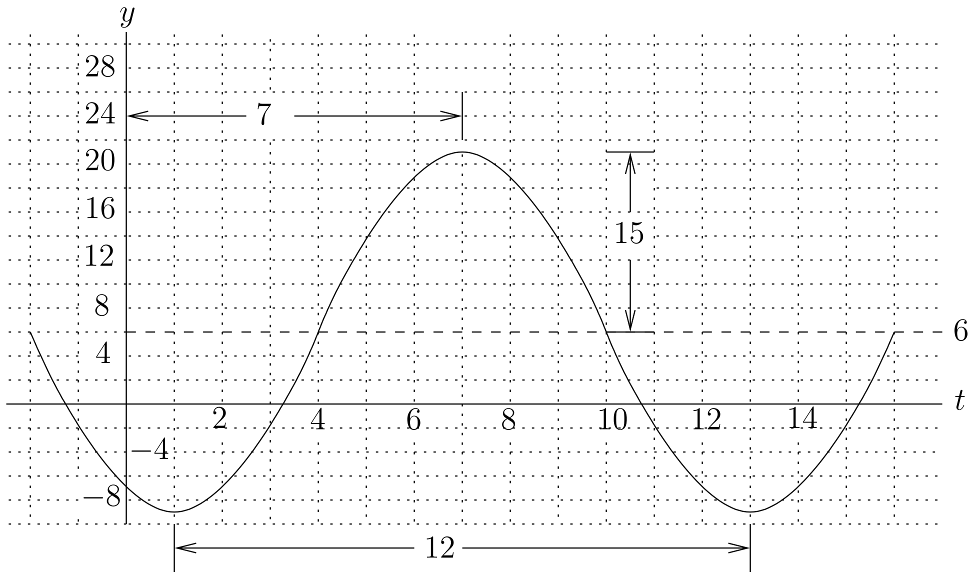
Question 2

4 points Over the course of a year, the city of Ottawa has its highest average monthly temperature of $21°\text{C}$ in August and its lowest monthly average of $-9°\text{C}$ in February. Assume that temperature varies sinusoidally over a period of one year. Find the parameters in the standard cosine description , i.e.,

$$f(x) = M + A\cos\left(\frac{2\pi}{P}(t - T)\right) ,$$

where t is in months, and $t = 0$ correspond to the month of January. Draw the graph of the function and identify the four parameters A, B, Φ, T in the graph. Give the names of the four parameters.

Solution:
The mean is $M = \dfrac{21 - 9}{2} = 6$, the amplitude is $A = \dfrac{21 + 9}{2} = 15$, the period is 12 months and the phase is $T = 7$ months. We get the following graph.



$$y = f(t)$$

Question 3

4 points **a**) Find the following limit without using a calculator

$$\lim_{x\to 3} \frac{|x-4|-1}{x^2-9} \ .$$

b) Does the following limit exist ? If yes, give the limit, if not, justify your answer. If you need a calculator to work out the answer, give at least 4 values of x that you tried.

$$\lim_{x\to -3} \frac{|x-4|-1}{x^2-9} \ .$$

Solution:

For $x < 4$, $|x-4| = 4-x$ and we have

$$\frac{|x-4|-1}{x^2-9} = \frac{4-x-1}{x^2-9} = \frac{3-x}{(x-3)(x+3)} = -\frac{1}{x+3} \ .$$

Since $f(x) = -1/(x+3)$ is continuous at $x = 3$, we have

$$\lim_{x\to 3} \frac{|x-4|-1}{x^2-9} = \lim_{x\to 3} \frac{-1}{x+3} = -\frac{1}{3+3} = -\frac{1}{6} \ .$$

b) As we saw in (**a**), for $x < 4$, $|x-4| = 4-x$ and we have

$$\frac{|x-4|-1}{x^2-9} = \frac{4-x-1}{x^2-9} = \frac{3-x}{(x-3)(x+3)} = -\frac{1}{x+3} \ .$$

However,

n	$-3-\frac{1}{n}$	$\frac{\lvert x-4\rvert-1}{x^2-9}=-\frac{1}{x+3}$
1	-4	1
2	-3.5	2
3	$-10/3$	3
$\vdots$	$\vdots$	$\vdots$
n	$-3-\frac{1}{n}$	n
$\downarrow$	$\downarrow$	$\downarrow$
∞	-3	∞

and

n	$-3+\frac{1}{n}$	$\frac{\lvert x-4\rvert-1}{x^2-9}=-\frac{1}{x+3}$
1	-2	-1
2	-2.5	-2
3	$-8/3$	-3
$\vdots$	$\vdots$	$\vdots$
n	$-3+\frac{1}{n}$	$-n$
$\downarrow$	$\downarrow$	$\downarrow$
∞	-3	$-\infty$

We conjecture that $\lim\limits_{x\to-3^+}\dfrac{|x-4|-1}{x^2-9}=-\infty$ and $\lim\limits_{x\to-3^-}\dfrac{|x-4|-1}{x^2-9}=\infty$. Since

$$\lim_{x\to-2^+}\frac{|x-4|-1}{x^2-9}\neq\lim_{x\to-2^+}\frac{|x-4|-1}{x^2-9},$$

the limit doesn't exist.

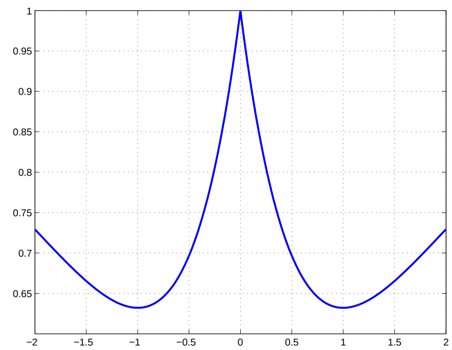
Question 4

4 points Use the definition of the derivative to calculate the derivative of the function $f(x)=(19x-9)^2+2009$

$$\begin{aligned}
f'(x)&=\lim_{h\to0}\frac{f(x+h)+-f(x)}{h}=\lim_{h\to0}\frac{\big((19(x+h)-9)^2+2009\big)-\big((19x-9)^2+2009\big)}{h}\\
&=\lim_{h\to0}\frac{(19x+19h-9)^2-(19x-9)^2}{h}\\
&=\lim_{h\to0}\frac{(19x-9)^2+38h(19x-9)+(19h)^2-(19x-9)^2}{h}\\
&=\lim_{h\to0}\frac{38h(19x-9)+(19h)^2}{h}=\lim_{h\to0}\big(38(19x-9)+19^2h\big)=38(19x-9)
\end{aligned}$$

Question 5

4 points Consider the graph of a function f below.



Find the critical points of f. Determine the intervals where f' is positive and the intervals where f' is negative.

Solution:
We have three critical points : at $x = -1$ and $x = 1$, where the derivative is null, and at $x = 0$, where the derivative doesn't exist.

The function is decreasing for $-2 \leq x < -1$ and $0 < x < 1$, and the fonctions is increasing for $-1 < x < 0$ and $1 < x \leq 2$.

Question 6

8 points The human metabolism will break down morphine in the blood stream to 80% of its original amount within a 4-hour period. A patient is given a dose of 7 mg of morphine every 4 hours. The DTDS for the amount of morphine in the patient's body is given by

$$M_{n+1} = 0.8M_t + d = 0.8M_n + 7 \quad , \quad n = 0, 1, 2, \ldots$$

where t is measured in 4-hour intervals and morphine is measured just after a new dose has been administered.

a) What is the updating function of the DTDS ?

b) Find the fixed point of the DTDS if there is one.

c) Find the general solution formula for the DTDS, i.e., $M_t = \ldots$

d) Graph the updating function and draw the cobwebbing, starting from $M_0 = 5$ for at least 4 steps.

e) Is the fixed point stable or unstable ?

f) The doctor wants to adjust the dose d so that the patient has 50 mg of morphine in the blood stream in the long run. What should the dose d be in that case ?

Solution:

a) The generatrice function is $f(x) = 0.8x + 7$.

b) p is an steady state if $f(p) = p$. Thus,

$$f(p) = p \Rightarrow p = 0.8p + 7 \Rightarrow 0.2p = 7 \Rightarrow p = 35 \ .$$

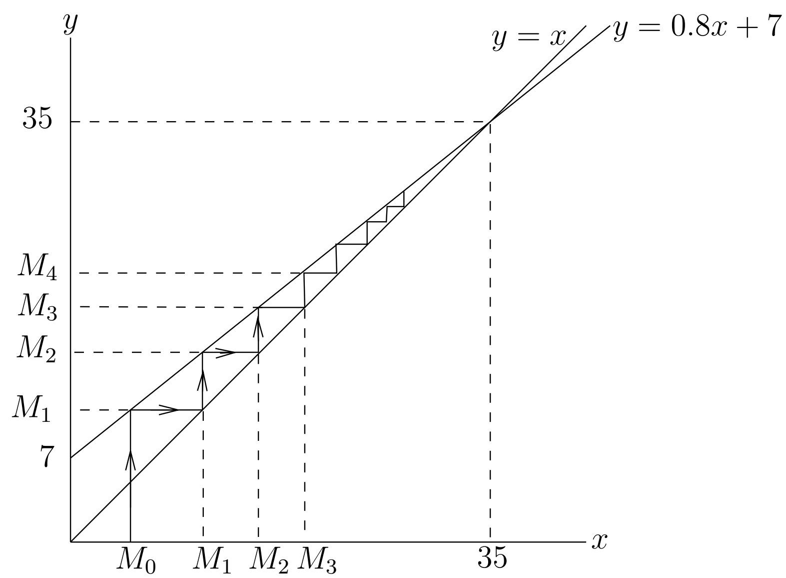
c) The general solution is

$$M_n = 0.8^n(M_0 - p) + p = 0.8^n(M_0 - 35) + 35 \quad , \quad n = 0, 1, 2, \ldots$$

Another form for the solution is

$$M_n = 0.8^n M_0 + 7(1 + 0.8 + 0.8^2 + 0.8^3 + ... + 0.8^{n-1}) \quad , \quad n = 1, 2, \ldots$$

d)



e) Since we have a linear system of the form $M_{n+1} = rM_n + b$ with $|r| < 1$, the equilibrium is stable. The stability is also obvious from cobwebbing as done in (**d**).

f) If we want the patient to have 50 mg of morphine in the blood in the long run, we must have an equilibrium point p such that $0.8p > 50$; namely, $p > 250/4 = 62.5$. We have $M_{n+1} = 0.8M_n + b$. The equilibrium point p is given by $p = 0.8p + b$. Thus, $p = 5b$. We need to choose $p = 5b > 62.5$; namely, $b > 62.5/5 = 12.5$ mg per four hours.

We need to choose a dose b larger than 12.5 mg per four hours.