Calcul différentiel et intégral pour les sciences de la vie I
MAT1730
Test 1

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Question 1

4 points Find the derivative of the following function : $f(x) = \dfrac{x^4 \ln(x)}{2x + 8e^x}$.

Solution:

We have $f(x) = g(x)/h(x)$, where $g(x) = x^4 \ln(x)$ and $h(x) = 2x + 8e^x$. Since

$$g'(x) = 4x^3 \ln(x) + x^4 \left(\frac{1}{x}\right) = 4x^3 \ln(x) + x^3 = x^3(4\ln(x) + 1) \ ,$$

we have

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{h^2(x)} = \frac{x^3(4\ln(x) + 1)(2x + 8e^x) - x^4 \ln(x)(2 + 8e^x)}{(2x + 8e^x)^2}$$

Question 2

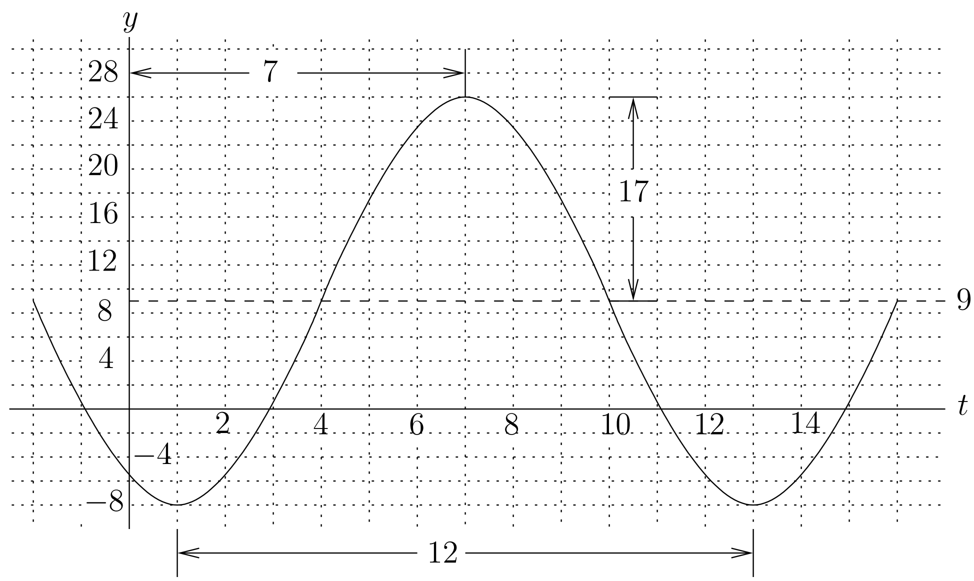
4 points Over the course of a year, the city of Ottawa has its highest average monthly temperature of 26°C in August and its lowest monthly average of −8°C in February. Assume that temperature varies sinusoidally over a period of one year. Find the parameters in the standard cosine description , i.e.,

$$f(x) = M + A\cos\left(\frac{2\pi}{P}(t - T)\right) \ ,$$

where t is in months, and $t = 0$ correspond to the month of January. Draw the graph of the function and identify the four parameters A, B, Φ, T in the graph. Give the names of the four parameters.

Solution:

The mean is $M = \dfrac{26 - 8}{2} = 9$, the amplitude is $A = \dfrac{26 + 8}{2} = 17$, the period is 12 months and the phase is $T = 7$ months. We get the following graph.



$y = f(t)$

Question 3

4 points **a**) Find the following limit without using a calculator

$$\lim_{x\to 2} \frac{|x-3|-1}{x^2-4} \ .$$

b) Does the following limit exist ? If yes, give the limit, if not, justify your answer. If you need a calculator to work out the answer, give at least 4 values of x that you tried.

$$\lim_{x\to -2} \frac{|x-3|-1}{x^2-4} \ .$$

Solution:

a) For $x < 3$, $|x-3| = 3-x$ and we have

$$\frac{|x-3|-1}{x^2-4} = \frac{3-x-1}{x^2-4} = \frac{2-x}{(x-2)(x+2)} = -\frac{1}{x+2} \ .$$

Since $f(x) = -1/(x+2)$ is continuous at $x = 2$, we have

$$\lim_{x\to 2} \frac{|x-3|-1}{x^2-4} = \lim_{x\to 2} \frac{-1}{x+2} = -\frac{1}{2+2} = -\frac{1}{4} \ .$$

b) As we saw in (**a**), for $x < 3$, $|x-3| = 3-x$ and we have

$$\frac{|x-3|-1}{x^2-4} = \frac{3-x-1}{x^2-4} = \frac{2-x}{(x-2)(x+2)} = -\frac{1}{x+2} \ .$$

However,

n	$-2-\frac{1}{n}$	$\frac{\lvert x-3\rvert-1}{x^2-4}=-\frac{1}{x+2}$
1	-3	1
2	-2.5	2
3	$-7/3$	3
$\vdots$	$\vdots$	$\vdots$
n	$-2-\frac{1}{n}$	n
$\downarrow$	$\downarrow$	$\downarrow$
∞	-2	∞

and

n	$-2+\frac{1}{n}$	$\frac{\lvert x-3\rvert-1}{x^2-4}=-\frac{1}{x+2}$
1	-1	-1
2	-1.5	-2
3	$-5/3$	-3
$\vdots$	$\vdots$	$\vdots$
n	$-2+\frac{1}{n}$	$-n$
$\downarrow$	$\downarrow$	$\downarrow$
∞	-2	$-\infty$

We conjecture that $\lim_{x\to -2^+}\dfrac{|x-3|-1}{x^2-4}=-\infty$ and $\lim_{x\to -2^-}\dfrac{|x-3|-1}{x^2-4}=\infty$. Since

$$\lim_{x\to -2^+}\frac{|x-3|-1}{x^2-4}\neq\lim_{x\to -2^+}\frac{|x-3|-1}{x^2-4},$$

the limit doesn't exist.

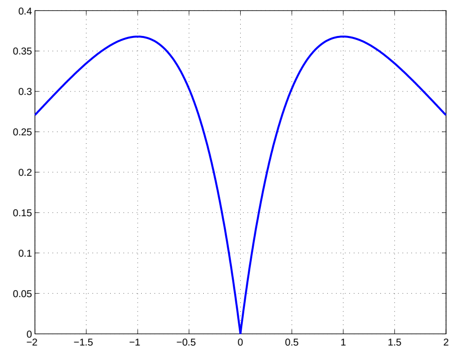
Question 4

4 points Use the definition of the derivative to calculate the derivative of the function $f(x)=(14x-10)^2+2009$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)+-f(x)}{h}=\lim_{h\to 0}\frac{\big((14(x+h)-10)^2+2009\big)-\big((14x-10)^2+2009\big)}{h}\\
&= \lim_{h\to 0}\frac{(14x+14h-10)^2-(14x-10)^2}{h}\\
&= \lim_{h\to 0}\frac{(14x-10)^2+28h(14x-10)+(14h)^2-(14x-10)^2}{h}\\
&= \lim_{h\to 0}\frac{28h(14x-10)+(14h)^2}{h}=\lim_{h\to 0}\big(28(14x-10)+14^2h\big)=28(14x-10)
\end{aligned}$$

Question 5

4 points Consider the graph of a function f below.



Find the critical points of f. Determine the intervals where f' is positive and the intervals where f' is negative.

Solution:
We have three critical points : at $x = -1$ and $x = 1$, where the derivative is null, and at $x = 0$, where the derivative doesn't exist.

The function is increasing for $-2 \leq x < -1$ and $0 < x < 1$, and the fonctions is decreasing for $-1 < x < 0$ and $1 < x \leq 2$.

Question 6

8 points The human metabolism will break down morphine in the blood stream to 70% of its original amount within a 4-hour period. A patient is given a dose of 9 mg of morphine every 4 hours. The DTDS for the amount of morphine in the patient's body is given by

$$M_{n+1} = 0.7M_t + d = 0.7M_n + 9 \quad , \quad n = 0, 1, 2, \ldots$$

where t is measured in 4-hour intervals and morphine is measured just after a new dose has been administered.

a) What is the updating function of the DTDS ?

b) Find the fixed point of the DTDS if there is one.

c) Find the general solution formula for the DTDS, i.e., $M_t = ...$

d) Graph the updating function and draw the cobwebbing, starting from $M_0 = 5$ for at least 4 steps.

e) Is the fixed point stable or unstable ?

f) The doctor wants to adjust the dose d so that the patient has 40 mg of morphine in the blood stream in the long run. What should the dose d be in that case ?

Solution:

a) The generatrice function is $f(x) = 0.7x + 9$.

b) p is an steady state if $f(p) = p$. Thus,

$$f(p) = p \Rightarrow p = 0.7p + 9 \Rightarrow 0.3p = 9 \Rightarrow p = 30 \ .$$

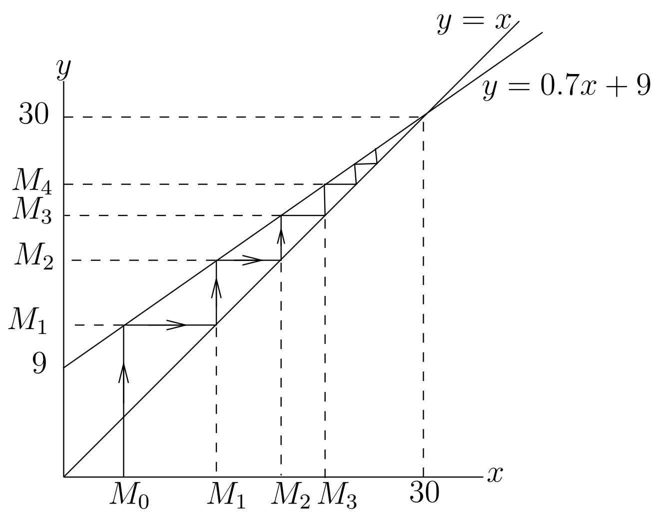
c) The general solution is

$$M_n = 0.7^n(M_0 - p) + p = 0.7^n(M_0 - 30) + 30 \quad , \quad n = 0, 1, 2, \ldots$$

Another form for the solution is

$$M_n = 0.7^n M_0 + 9(1 + 0.7 + 0.7^2 + 0.7^3 + ... + 0.7^{n-1}) \quad , \quad n = 1, 2, \ldots$$

d)



e) Since we have a linear system of the form $M_{n+1} = rM_n + b$ with $|r| < 1$, the equilibrium is stable. The stability is also obvious from cobwebbing as done in (**d**).

f) If we want the patient to have 40 mg of morphine in the blood in the long run, we must have an equilibrium point p such that $0.7p > 40$; namely, $p > 400/7 = 57.\overline{142857}$. We have $M_{n+1} = 0.7M_n + b$. The equilibrium point p is given by $p = 0.7p + b$. Thus, $p = 10b/3$. We need to choose $p = 10b/3 > 400/7$; namely, $b > 120/7 = 17.\overline{142857}$ mg per four hours.

We need to choose a dose b larger than $17.\overline{142857}$ mg per four hours.