Calculus for the Life Science I
MAT1330A , MAT1330B, MAT1330E
Assignment 7

Question 1
We consider the function $g(x) = e^x - 2 - \cos(x)$.

a) Why is there a solution of $g(x) = 0$ between 0 and 1 ?

b) Use Newton's method with $x_0 = 0$ to find this root. Stop when the difference between two consecutive iterations is less than 10^{-4}.

Solution:
a) Since g is a continuous function and, $g(0) = -2 < 0$ and $g(1) = 0.1779795\ldots > 0$, we may use the Intermediate Value Theorem to conclude that there exists c between 0 and 1 where $g(c) = 0$.

b) Since $g'(x) = e^x + \sin(x)$, we have

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} = x_n - \frac{e^{x_n} - 2 - \cos(x_n)}{e^{x_n} + \sin(x_n)} \quad , \quad n = 0, 1, 2, \ldots$$

With $x_0 = 0$, we get

$$x_1 = 0 - \frac{e^0 - 2 - \cos(0)}{e^0 + \sin(0)} = 2 \quad , \quad x_2 = 2 - \frac{e^2 - 2 - \cos(2)}{e^2 + \sin(2)} = 1.30043919 \quad , \ldots$$

We get the following results.

n	x_n	$\lvert x_{n+1} - x_n \rvert$
0	0	
1	2	2
2	1.30043919	$0.69956081 \not< 10^{-4}$
3	0.99753544	$0.30290375 \not< 10^{-4}$
4	0.94989209	$0.04764334 \not< 10^{-4}$
5	0.94881530	$0.00107680 \not< 10^{-4}$
6	0.94881476	$0.00000054 < 10^{-4}$

The solution is approximatively $x_6 = 0.94881476\ldots$

Question 2
Compute the following integrals.

a) $\displaystyle\int \frac{x^2 - 1}{x^2 + 1}\,\mathrm{d}x$ **b)** $\displaystyle\int \sqrt{e^x}\,\mathrm{d}x$ **c)** $\displaystyle\int \frac{(1 + 3\ln(x))^{50}}{x}\,\mathrm{d}x$

d) $\displaystyle\int \frac{x^{1/6}}{1 + x^{1/3}}\,\mathrm{d}x$ **e)** $\displaystyle\int (x^2 + 1)\sin(3x)\,\mathrm{d}x$ **f)** $\displaystyle\int e^{\cos(x)}\cos(x)\sin(x)\,\mathrm{d}x$

g) $\displaystyle\int (x^3 + 2)\ln(x)\,\mathrm{d}x$

Solution:

a) We first divide the quotient $\dfrac{x^2-1}{x^2+1}$ to get $\dfrac{x^2-1}{x^2+1} = 1 - \dfrac{2}{x^2+1}$. Hence,

$$\int \frac{x^2-1}{x^2+1}\,\mathrm{d}x = \int \left(1 - \frac{2}{x^2+1}\right)\mathrm{d}x = \int \mathrm{d}x - 2\int \frac{1}{x^2+1}\,\mathrm{d}x = x - 2\arctan(x) + C \ .$$

b) It suffices to note that

$$\int \sqrt{e^x}\,\mathrm{d}x = \int e^{x/2}\,\mathrm{d}x \ .$$

Let $u = x/2$. Then, $2\,\mathrm{d}u = \mathrm{d}x$ and

$$\int \sqrt{e^x}\,\mathrm{d}x = 2\int e^u\,\mathrm{d}u = 2e^u + C = 2e^{x/2} + C \ .$$

c) Let $u = 1 + 3\ln(x)$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{3}{x} \Rightarrow \frac{1}{3}\,\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x \ .$$

Hence,

$$\int \frac{(1+3\ln(x))^{50}}{x}\,\mathrm{d}x = \frac{1}{3}\int u^{50}\,\mathrm{d}u = \frac{1}{153}u^{51}\Big|_{u=1+3\ln(x)} + C = \frac{1}{153}(1+3\ln(x))^{51} + C \ .$$

d) Let $u = x^{1/6}$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{6}x^{-5/6} \Rightarrow 6x^{5/6}\,\mathrm{d}u = \mathrm{d}x \Rightarrow 6u^5\,\mathrm{d}u = \mathrm{d}x \ .$$

We get

$$\int \frac{x^{1/6}}{1+x^{1/3}}\,\mathrm{d}x = 6\int \frac{u^6}{1+u^2}\,\mathrm{d}u \ .$$

If we divide $\dfrac{u^6}{1+u^2}$, we get $\dfrac{u^6}{1+u^2} = u^4 - u^2 + 1 - \dfrac{1}{1+u^2}$. Hence,

$$\begin{aligned}
\int \frac{x^{1/6}}{1+x^{1/3}}\,\mathrm{d}x &= 6\int \left(u^4 - u^2 + 1 - \frac{1}{1+u^2}\right)\mathrm{d}u \\
&= 6\int u^4\,\mathrm{d}u - 6\int u^2\,\mathrm{d}u + 6\int \mathrm{d}u - 6\int \frac{1}{1+u^2}\,\mathrm{d}u \\
&= \frac{6u^5}{5} - 2u^3 + 6u - 6\arctan(u) + C \\
&= \frac{6x^{5/6}}{5} - 2x^{1/2} + 6x^{1/6} - 6\arctan(x^{1/6}) + C
\end{aligned}$$

e) We use integration by parts.

$$\int (x^2+1)\sin(3x)\,\mathrm{d}x = \int f(x)g'(x)\,\mathrm{d}x$$

with $f(x) = x^2 + 1$ and $g'(x) = \sin(3x)$. Thus, $f'(x) = 2x$ and $g(x) = -\dfrac{1}{3}\cos(3x)$. Hence,

$$\begin{aligned}\int (x^2+1)\sin(3x)\,\mathrm{d}x &= \int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x \\ &= -\frac{1}{3}(x^2+1)\cos(3x) + \frac{2}{3}\int x\cos(3x)\,\mathrm{d}x \ .\end{aligned}$$

We again use integration by parts to compute $\int x\cos(3x)\,\mathrm{d}x$.

$$\int x\cos(3x)\,\mathrm{d}x = \int f(x)g'(x)\,\mathrm{d}x$$

with $f(x) = x$ and $g'(x) = \cos(3x)$. Thus, $f'(x) = 1$ and $g(x) = \dfrac{1}{3}\sin(3x)$. Hence,

$$\begin{aligned}\int x\cos(3x)\,\mathrm{d}x &= \int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x \\ &= \frac{1}{3}x\sin(3x) - \frac{1}{3}\int \sin(3x)\,\mathrm{d}x = \frac{1}{3}x\sin(3x) + \frac{1}{9}\cos(3x) + D \ .\end{aligned}$$

We finally get

$$\begin{aligned}\int (x^2+1)\sin(3x)\,\mathrm{d}x &= -\frac{1}{3}(x^2+1)\cos(3x) + \frac{2}{3}\left(\frac{1}{3}x\sin(3x) + \frac{1}{9}\cos(3x) + D\right) \\ &= -\frac{1}{3}(x^2+1)\cos(3x) + \frac{2}{9}x\sin(3x) + \frac{2}{27}\cos(3x) + C \ .\end{aligned}$$

where $C = 2D/3$.

f) Let $u = \cos(x)$. We have $\dfrac{\mathrm{d}u}{\mathrm{d}x} = -\sin(x)\,\mathrm{d}x$. Hence,

$$\int e^{\cos(x)}\cos(x)\sin(x)\,\mathrm{d}x = -\int ue^u\,\mathrm{d}u \ .$$

We use integration by parts.

$$\int ue^u\,\mathrm{d}x = \int f(u)g'(u)\,\mathrm{d}u \ ,$$

where $f(u) = u$ and $g'(u) = e^u$. Thus, $f'(u) = 1$ and $g(u) = e^u$. We get

$$\begin{aligned}I &= \int ue^u\,\mathrm{d}x = \int f(u)g'(u)\,\mathrm{d}u = f(u)g(u) - \int f'(u)g(u)\,\mathrm{d}u \\ &= ue^u - \int e^u\,\mathrm{d}u = ue^u - e^u + C\end{aligned}$$

Finally,

$$\int e^{\cos(x)}\cos(x)\sin(x)\,\mathrm{d}x = -\left(ue^u - e^u + C\right) = -\cos(x)e^{\cos(x)} + e^{\cos(x)} - C \ .$$

g) We use integration by parts.

$$\int (x^3+2)\ln(x)\,\mathrm{d}x = \int f(x)g'(x)\,\mathrm{d}x \ ,$$

where $f(x)=\ln(x)$ and $g'(x)=x^3+2$. Thus, $f'(x)=1/x$ and $g(x)=\dfrac{x^4}{4}+2x$. Hence,

$$\begin{aligned}
&\int (x^3+2)\ln(x)\,\mathrm{d}x = \int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x \\
&= \left(\frac{x^4}{4}+2x\right)\ln(x) - \int \left(\frac{x^4}{4}+2x\right)\frac{1}{x}\,\mathrm{d}x = \left(\frac{x^4}{4}+2x\right)\ln(x) - \int \left(\frac{x^3}{4}+2\right)\mathrm{d}x \\
&= \left(\frac{x^4}{4}+2x\right)\ln(x) - \left(\frac{x^4}{16}+2x\right) + C = \frac{x^4}{4}\left(\ln(x)-\frac{1}{4}\right) + 2x(\ln(x)-1) + C
\end{aligned}$$

Question 3

Let $p(t)$ be the position (in meters) of an object along a straight line at time t (in minutes). We know that the velocity of the object at time t is given by $p'(t)=t\sin(t)$. Find the position of the object after 15 minutes if initially the object is at 10 meters from the origin.

Solution:

We need to find the function p. The function p is a primitive of $t\sin(t)$. We use the method of integration by parts to evaluate the integral $\displaystyle\int t\sin(t)\,\mathrm{d}t$. We have

$$\int t\sin(t)\,\mathrm{d}t = \int f(t)g'(t)\,\mathrm{d}t \ ,$$

where $f(t)=t$ and $g'(t)=\sin(t)$. Thus $f'(t)=1$ and $g(t)=-\cos(t)$. Hence,

$$\begin{aligned}
\int t\sin(t)\,\mathrm{d}t &= \int f(t)g'(t)\,\mathrm{d}t = f(t)g(t) - \int f'(t)g(t)\,\mathrm{d}t \\
&= -t\cos(t) + \int \cos(t)\,\mathrm{d}t = -t\cos(t) + \sin(t) + C \ .
\end{aligned}$$

We have $p(t) = -t\cos(t)+\sin(t)+C$ for a constant C which is determined by the initial condition $p(0)=C=10$. Thus

$$p(t) = -t\cos(t)+\sin(t)+10$$

and $p(15) = -15\cos(15)+\sin(15)+10 \approx 22.0456065$ m.