Calculus for the Life Science I
MAT1330A , MAT1330B, MAT1330E
Assignment 6

Question 1
An animal species is described by the discrete dynamical system

$$N_{i+1} = 1.5N_i(1 - N_i) - hN_i \quad , \quad i = 0, 1, 2, 3, \ldots ,$$

where N_i is the fraction of the maximal population after i years. h is the harvesting effort of the predators.

a) Find the equilibrium points. One of these points will depend of h.

b) Given the largest interval for h such that the equilibrium points in (**a**) have a biological meaning.

c) Find the equilibrium harvest R for this species as a function of h.

d) Determine the harvesting effort h that will maximize the equilibrium harvest.

e) Give the maximal equilibrium harvest.

f) Is the maximal equilibrium harvest stable or unstable ?

Solution:
a) We seek P such that $P = 1.5P(1 - P) - hP$. $P = 0$ is a solution. It is the first equilibrium point. If $P \neq 0$, we may divide both sides of $P = 1.5P(1 - P) - hP$ to get $1 = 1.5(1 - P) - h$. Solving for P yields a second equilibrium point ; namely, $P = P(h) = (1 - 2h)/3$.

b) We must have $h \geq 0$. Moreover, for the equilibrium point $P(h)$ to be positive, we must have $h \leq 1/2$. The maximal interval is $[0, 1/2]$.

c) The equilibrium harvest is $R(h) = hP(h) = \dfrac{h(1 - 2h)}{3}$.

d) R is a continuous function on the closed interval $[0, 1/2]$. We may use the Extreme Value Theorem to find the global maximum.

Since $R'(x) = \dfrac{1}{3} - \dfrac{4h}{3}$, we get $h = 1/4$ from $R'(x) = 0$. We have $R(0) = R(1/2) = 0$ and $R(1/4) = 1/24$. Thus, the equilibrium harvest is given by $h = 1/4$.

e) The equilibrium harvest is $R(1/4) = 1/24$.

f) The equilibrium harvest is stable. The updating function of this discrete dynamical system for $h = 1/4$ is

$$f(x) = 1.5x(1 - x) - (1/4)x = -1.5x^2 + 1.25x$$

and the equilibrium point is

$$P = P(1/4) = \frac{1-2(1/4)}{3} = \frac{1}{6} \ .$$

Hence, $f'(x) = -3x + 1.25$ and $f'(1/6) = -3(1/6) + 1.25 = 0.75$. Since, $|f'(1/6)| < 1$, the equilibrium point $1/6$ is stable. Thus, the equilibrium harvest is stable.

Question 2

The Law of mass action states that the rate of a chemical reaction between reactants is proportional to the product of the concentrations of the reactants.

Consider the chemical reaction

$$A + 2B \rightarrow AB_2$$

between two chemical reactants A and B. If a is the initial concentration of the reactant A and b is the initial concentration of the reactant B, then we get from the law of mass action that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k(a-x)(b-2x)^2 \ ,$$

where x is the concentration of the product AB_2 of the chemical reaction and k is a constant of proportionality. The concentration is the number of molecules per unit of volume. We have that $0 \leq x \leq \min\{a, b/2\}$ (the minimum of a and $b/2$) because the chemical reaction stops when one of the reactant is exhausted. We assume in the model above that no reactant is added to the system after the beginning of the chemical reaction.

If $a = 2$ and $b = 3$, find the concentration x of the product of the reaction when the rate of the reaction is the fastest. Note that the answer is independent of the value of k.

Solution:

For $a = 2$ and $b = 3$, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k(2-x)(3-2x)^2 \ .$$

We seek the value of x where $k(a-x)(b-2x)^2$ reaches is global maximum. Since we aren't looking for the global maximum but for the value where the global maximum is reached, the value of $k > 0$ doesn't matter. It suffices to find the value of x where $f(x) = (2-x)(3-2x)^2$ reaches its global maximum on the closed interval $[0, 3/2]$. We may use the Extreme Value Theorem.

We have $f'(x) = -(3-2x)^2 - 4(2-x)(3-2x) = (3-2x)(-11+6x)$. The critical points are given by $f'(x) = 0$ only. We find $x = 3/2$ and $x = 11/6$. We ignore $x = 11/6$ because $11/6 > 3/2$.

Since $f(0) = 18$ and $f(3/2) = 0$, The global maximum is reached at $x = 0$. The maximal rate of the reaction is at the beginning of the reaction.

Question 3

Use a cubic Taylor polynomial to approximate the value of $\ln(1.01)$.

Solution:

Let $f(x) = \ln(x)$. Since $f'(x) = 1/x$, $f''(x) = -1/x^2$ and $f'''(x) = 2/x^3$, The Taylor polynomial of degree three of $f(x) = \ln(x)$ for x near $x = 1$ is

$$p(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 + \frac{1}{3!}f'''(1)(x-1)^3$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 \ .$$

Thus,

$$\ln(1.01) \approx p(1.01) = 0.01 - \frac{0.01^2}{2} + \frac{0.01^3}{3} = 0.009950\overline{3} \ .$$

This is a good approximation of $\ln(1.01) = 0.009950330\ldots$.

Question 4

For each of the limits below, determine if it is possible to use the Hospital's rule to evaluate the limit and, if it is possible, evaluate the limit.

a) $\displaystyle\lim_{x\to 0} \frac{4-\sqrt{3x^2+16}}{3x}$ **b)** $\displaystyle\lim_{x\to\infty} \frac{e^x+1}{x^2}$

c) $\displaystyle\lim_{x\to 0} x^{-1/3}\sin(x)$ **d)** $\displaystyle\lim_{x\to 1} \frac{2-x}{\ln(2-x)}$

Solution:

a) We have the indeterminate form (0/0). We may use the Hospital rule. We have

$$\lim_{x\to 0} \frac{4-\sqrt{3x^2+16}}{3x} = \lim_{x\to 0} \frac{-3x(3x^2+16)^{-1/2}}{3} = \lim_{x\to 0} \frac{-x}{\sqrt{3x^2+16}} = \frac{0}{4} = 0 \ .$$

The first equality comes from the Hospital rule. The other equalities comes from simple algebraic manipulations.

b) We have the indeterminate form (∞/∞). We may use the Hospital rule. We have

$$\lim_{t\to\infty} \frac{e^x+1}{x^2} = \lim_{t\to\infty} \frac{e^x}{2x} \ .$$

The limit on the right hand side of the equality is an indeterminate form (∞/∞). We may use the Hospital rule.

$$\lim_{t\to\infty} \frac{e^x+1}{x^2} = \lim_{t\to\infty} \frac{e^x}{2x} = \lim_{t\to\infty} \frac{e^x}{2} = \infty \ .$$

c) Since

$$\lim_{t\to 0} x^{-1/3}\sin(x) = \lim_{t\to 0} \frac{\sin(x)}{x^{1/3}} \ ,$$

we have an indeterminate form (0/0). We may use the Hospital rule to get :

$$\lim_{t\to 0} x^{-1/3}\sin(x) = \lim_{t\to 0} \frac{\cos(x)}{x^{-2/3}/3} = \lim_{t\to\infty} 3x^{2/3}\cos(x) = 0$$

because $\cos(x) \to \cos(0) = 1$ for $x \to 0$ and $x^{2/3} \to 0$ for $x \to 0$.

d) Even if $\ln(2-x) \to \ln(1) = 0$ as $x \to 1$, we cannot use the Hospital rule because $(2-x) \to 1 \neq 0$ as $x \to 1$. We don't have the indeterminate form (0/0). We must fall back on the table of values to show that

$$\lim_{x\to 1^-} \frac{2-x}{\ln(2-x)} = +\infty \qquad \text{and} \qquad \lim_{x\to 1^+} \frac{2-x}{\ln(2-x)} = -\infty \ .$$

Thus, the limit doesn't exist.