Solutions

MAT 1322 A **Assignment 5** (Due Wed. March 30th at 8:30) **Student Number:**

1. Starting from the Maclaurin series for $\frac{1}{1-x}$, find the Maclaurin series of (i) $\ln(1+x^2)$ and (ii) $\int \ln(1+x^2)\,dx$ and give their intervals of convergence.

Work:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (\text{for } -1<x<1)$$

then $$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad (\text{also for } -1<x<1)$$

so $$\frac{2x}{1+x^2} = \sum_{n=0}^{\infty} 2(-1)^n x^{2n+1} \quad (\text{for } -1<x<1)$$

thus $$\ln(1+x^2) = \int \frac{2x}{1+x^2}\,dx = \int \sum_{n=0}^{\infty} 2(-1)^n x^{2n+1}\,dx$$
$$= C + \sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n+2}}{2n+2} \quad (\text{for } -1<x<1)$$

but $\ln(1)=0 \Rightarrow C=0$

so $$\ln(1+x^2) = \sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n+2}}{2n+2}$$

check endpoints: if $x=\pm 1$ $\sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n+2}}{2n+2} = \sum_{n=0}^{\infty} \frac{2(-1)^n}{2n+2}$ converges (AST)

thus interval is $-1 \le x \le 1$

then $$\int \ln(1+x^2)\,dx = \int \sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n+2}}{2n+2}\,dx = C + \sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n+3}}{(2n+3)(2n+2)}$$
$(\text{for } -1 \le x \le 1)$

Answers:

i, $\ln(1+x^2) = \sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n+2}}{2n+2}$, $-1 \le x \le 1$

ii $\int \ln(1+x^2)\,dx = C + \sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n+3}}{(2n+3)(2n+2)}$, $-1 \le x \le 1$

2. Find the Maclaurin series of $f(x) = (1+x^2)^{2/3}$.

Work:

Binomial with $k = 2/3$

$$(1+x^2)^{2/3} = \sum_{n=0}^{\infty} \binom{2/3}{n} x^{2n}$$

$$\binom{2/3}{n} = \frac{(2/3)(-1/3)(-4/3)(-7/3)(-10/3)\cdots(2/3-n+1)}{n!}$$

$$= \frac{(2/3)(-1/3)(-4/3)(-7/3)(-10/3)\cdots(-1/3)(3n-5)}{n!}$$

$$= \frac{2(-1)^{n-1}\,(1\cdot 4\cdot 7\cdots(3n-5))}{3^n\, n!}$$

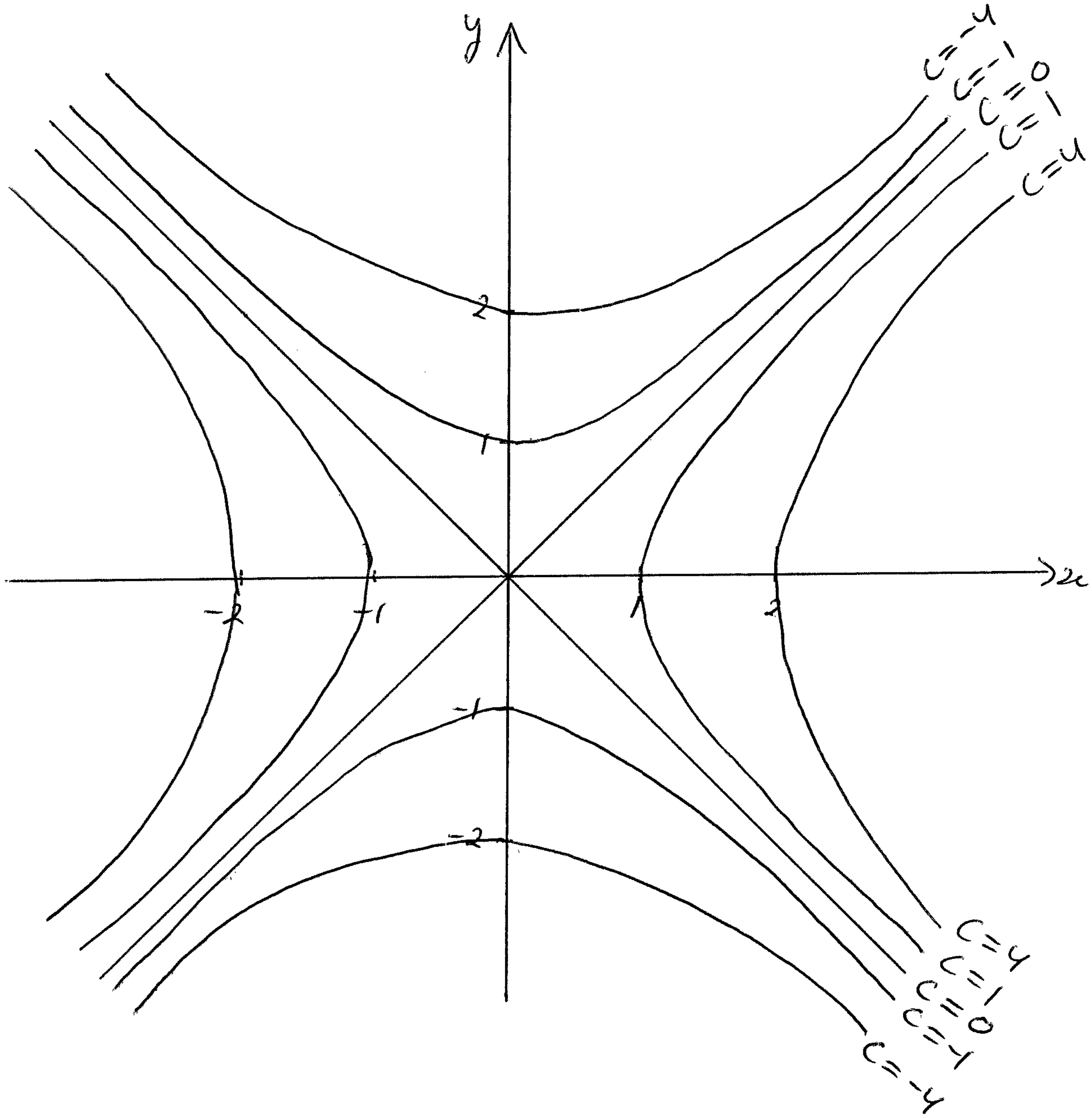
the formula makes sense for $n \geq 2$

$$\binom{2/3}{0} = 1 \quad , \quad \binom{2/3}{1} = 2/3$$

Answer: $$\boxed{(1+x^2)^{2/3} = 1 + \frac{2}{3}x^2 + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}(1\cdot 4\cdot 7\cdots(3n-5))}{3^n\, n!} x^{2n}}$$

3. Sketch, on the same graph, the level curves of $x^2 - y^2 = C$ for $C = -4, -1, 0, 1, 4$. Indicate the scale on your axes and label the contours.

Answer:



4. Find all first and second partial derivatives of the functions.

(a) $f(x, y) = 2x^2y^3 + 3x - \ln y$ (b) $g(r, \theta) = r^2 \cos(3\theta) - e^{3r}$ (c) $f(x, t) = 3x\cos(2t^2 - 4)$

Work and Answers:

a)

$$\boxed{\begin{aligned} f_x &= 4xy^3 + 3 \\ f_y &= 6x^2y^2 - 1/y \\ f_{xx} &= 4y^3 \\ f_{xy} &= 12xy^2 \\ f_{yy} &= 12x^2y + 1/y^2 \\ f_{yx} &= 12xy^2 \end{aligned}}$$

b)

$$\boxed{\begin{aligned} g_r &= 2r\cos(3\theta) - 3e^{3r} \\ g_\theta &= -3r^2\sin(3\theta) \\ g_{rr} &= 2\cos(3\theta) - 9e^{3r} \\ g_{r\theta} &= -6r\sin(3\theta) \\ g_{\theta\theta} &= -9r^2\cos(3\theta) \\ g_{\theta r} &= -6r\sin(3\theta) \end{aligned}}$$

c)

$$\boxed{\begin{aligned} f_x &= 3\cos(2t^2 - 4) \\ f_t &= -12xt\sin(2t^2 - 4) \\ f_{xx} &= 0 \\ f_{xt} &= -12t\sin(2t^2 - 4) \\ f_{tt} &= -12x\sin(2t^2 - 4) - 48xt^2\cos(2t^2 - 4) \\ f_{tx} &= -12t\sin(2t^2 - 4) \end{aligned}}$$