Solutions

MAT 1322 A **Assignment 4** (Due Wed. March 9th at 8:30) **Student Number:**

1. Determine whether the series converges or diverges. If it converges, find the sum.

(a) $\sum_{n=1}^{\infty} \frac{3^n + 2^n}{6^{n+2}}$

Work:

a) $\sum_{n=1}^{\infty} \frac{3^n+2^n}{6^{n+2}}$

$= \sum_{n=1}^{\infty} \left(\left(\frac{3}{6^3}\right)\left(\frac{3}{6}\right)^{n-1} + \left(\frac{2}{6^3}\right)\left(\frac{2}{6}\right)^{n-1} \right)$

$= \sum_{n=1}^{\infty} \left(\frac{1}{72}\right)\left(\frac{1}{2}\right)^{n-1} + \sum_{n=1}^{\infty} \left(\frac{1}{108}\right)\left(\frac{1}{3}\right)^{n-1}$

$= \frac{1/72}{1-1/2} + \frac{1/108}{1-1/3}$ (convergent geometric series)

$= \left(\frac{1}{72}\right)(2) + \left(\frac{1}{108}\right)\left(\frac{3}{2}\right)$

$= \frac{2}{72} + \frac{1}{72}$

$= \frac{3}{72}$

(b) $\sum_{n=1}^{\infty} \frac{2}{n(n+4)}$

b) use partial fractions to see that

$\frac{2}{n(n+4)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+4}\right)$

so $\sum_{n=1}^{\infty} \frac{2}{n(n+4)} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+4}\right)$

(ie telescoping series)

the nth partial sum is $S_n = \sum_{k=1}^{n} \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+4}\right)$

$= \frac{1}{2}\Big[1 - \cancel{\frac{1}{5}} + \frac{1}{2} - \cancel{\frac{1}{6}}$
$+ \frac{1}{3} - \cancel{\frac{1}{7}} + \frac{1}{4} - \cancel{\frac{1}{8}}$
$+ \cancel{\frac{1}{5}} - \frac{1}{9} + \cancel{\frac{1}{6}} - \cancel{\frac{1}{10}} +$
$\cdots + \cancel{\frac{1}{n-1}} - \frac{1}{n+3} + \frac{1}{n} - \frac{1}{n+4}\Big]$

$= \frac{1}{2}\Big[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$
$- \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3} - \frac{1}{n+4}\Big]$

so $\lim_{n\to\infty} S_n = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)$

(c) $\sum_{n=1}^{\infty} \frac{n^2 + \sin(n)}{n^2+1}$

c) here $a_n = \frac{n^2 + \sin(n)}{n^2+1}$

so $\lim_{n\to\infty} a_n \neq 0$

Answers: (a) converges to $1/24$ (b) converges to $25/24$ (c) diverges

2. Determine whether the series converges or diverges. Verify that the test used is applicable.

(a) $\sum_{n=1}^{\infty} \frac{4+\sin(n)}{n^3+2}$

(b) $\sum_{n=3}^{\infty} \frac{2}{n\ln(n)}$

Work:

a) for all $n \geq 1$, $-1 \leq \sin(n) \leq 1$

so $3 \leq 4+\sin(n) \leq 5$

$\left(\text{thus} \quad \frac{4+\sin(n)}{n^3+2} > 0 \right)$

also $\frac{1}{n^3+2} < \frac{1}{n^3}$ for $n \geq 1$

so $\frac{4+\sin(n)}{n^3+2} \leq \frac{5}{n^3+2} < \frac{5}{n^3}$

$\therefore \sum_{n=1}^{\infty} \frac{4+\sin(n)}{n^3+2} < \sum_{n=1}^{\infty} \frac{5}{n^3}$ which

is known to converge
(p series with $p=3>1$)

b) let $f(x) = \frac{2}{x\ln x}$

for all $x \geq 3$, $f(x)$ is positive, decreasing and continuous.

$$\int_3^{\infty} \frac{2}{x\ln x}\,dx = \lim_{b\to\infty} \int_3^b \frac{2}{x\ln x}\,dx$$

$$= \lim_{b\to\infty} 2\ln(\ln x)\Big|_3^b$$

$$= \lim_{b\to\infty} 2\left(\ln(\ln b) - \ln(\ln 3)\right)$$

$$= \infty$$

$\therefore$ improper integral diverges

Answers: (a) $\boxed{\text{converges (Comparison Test)}}$

(b) $\boxed{\text{diverges (Integral Test)}}$

3. Determine whether the series is convergent or divergent. If it converges, is it absolutely convergent?

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^3}{(n+2)!}$

(b) $\sum_{n=3}^{\infty} \frac{(-1)^n}{n \ln(n)}$

Work:

a) use the Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^n (n+1)^3}{(n+3)!} \Big/ \frac{(-1)^{n-1} n^3}{(n+2)!} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \frac{(n+2)!}{(n+3)!}$$

$$= \lim_{n\to\infty} \frac{(n+1)^3}{n^3 (n+3)}$$

$$= 0 < 1$$

b) have alternating series where

$$b_n = \frac{1}{n \ln(n)}$$

so $b_{n+1} < b_n$ and $b_n \to 0$ as $n \to \infty$

so it converges (AST)

<u>but</u> $\sum_{n=3}^{\infty} \frac{1}{n \ln(n)}$ will diverge

(see #2(b))

Answers: (a) $\boxed{\text{Converges absolutely (Ratio Test)}}$

(b) $\boxed{\text{Converges conditionally (AST \& Integral)}}$

4. Find the radius and interval of convergence of the power series.

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n (x-3)^n}{(n+1)!}$

Work:

(Centre is $a=3$)

a) $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-3)^{n+1}}{(n+2)!} \Big/ \frac{(-1)^n (x-3)^n}{(n+1)!} \right|$

$= \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}(n+1)!}{(x-3)^n (n+2)!} \right|$

$= |x-3| \lim_{n\to\infty} \frac{1}{n+2}$

$= 0$ (for all x)

So converges for all x

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{2^n n^2}$

(Centre is $a=-2$)

b) $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x+2)^{n+1}}{2^{n+1}(n+1)^2} \Big/ \frac{(-1)^n (x+2)^n}{2^n n^2} \right|$

$= \lim_{n\to\infty} \left| \frac{(x+2)^{n+1}\, 2^n\, n^2}{(x+2)^n\, 2^{n+1}(n+1)^2} \right|$

$= \frac{|x+2|}{2} \lim_{n\to\infty} \frac{n^2}{(n+1)^2}$

$= \frac{|x+2|}{2}$

need $\frac{|x+2|}{2} < 1$ for convergence

or $|x+2| < 2$ $\quad R=2$, interval $-4 < x < 0$

Check endpoints:

if $x=-4$ $\quad \sum_{n=1}^{\infty} \frac{(-1)^n(-2)^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ (converges)

if $x=0$ $\quad \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ (converges)

Answers: (a) $\boxed{R=\infty \text{, interval } (-\infty, \infty)}$

(b) $\boxed{R=2 \text{, interval } -4 \le x \le 0}$