Solutions

MAT 1322 A **Assignment 3** (Due Wed. Feb. 16th at 8:30) **Student Number:**

1. Consider the initial value problem $y' = -4xy$, $y(0) = 3$.
(a) Use Euler's Method to estimate $y(1)$ to 4 decimal places, with 10 steps.
(b) What is the true solution of the initial value problem?
(c) What is the true value of $y(1)$ to 4 decimal places?
(d) What is the error of the approximation in (a)?

Solution: **Answers:** (a) $y(1) \approx \boxed{0.3731}$ (b) $y(x) = \boxed{3e^{-2x^2}}$

(c) $y(1) = \boxed{0.4060}$ (d) $\epsilon = \boxed{0.0329}$

a) $y' = -4xy$, $y(0) = 3$, 10 steps $\Rightarrow$ $h = \frac{1-0}{10} = 0.1$

so $f(x,y) = -4xy$, $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, etc..., $y_0 = 3$

$y_{n+1} = y_n + hf(x_n, y_n) = y_n + (0.1)(-4)x_n y_n = y_n(1 - 0.4x_n)$

$y_1 = y_0(1 - 0.4x_0) = 3(1 - 0.4(0)) = 3$

$y_2 = y_1(1 - 0.4x_1) = 3(1 - 0.4(0.1)) = 2.88$

$y_3 = y_2(1 - 0.4x_2) = 2.88(1 - 0.4(0.2)) = 2.6496$

$y_4 = y_3(1 - 0.4x_3) = 2.6496(1 - 0.4(0.3)) = 2.3316$

$y_5 = y_4(1 - 0.4x_4) = 2.3316(1 - 0.4(0.4)) = 1.9585$

$y_6 = y_5(1 - 0.4x_5) = 1.9585(1 - 0.4(0.5)) = 1.5668$

$y_7 = y_6(1 - 0.4x_6) = 1.5668(1 - 0.4(0.6)) = 1.1908$

$y_8 = y_7(1 - 0.4x_7) = 1.1908(1 - 0.4(0.7)) = 0.8574$

$y_9 = y_8(1 - 0.4x_8) = 0.8574(1 - 0.4(0.8)) = 0.5830$

$y_{10} = y_9(1 - 0.4x_9) = 0.5830(1 - 0.4(0.9)) = 0.3731 \approx y(1)$

b) $\frac{dy}{dx} = -4xy$

separate $\frac{dy}{y} = -4x\,dx$

integrate $\int \frac{dy}{y} = \int -4x\,dx + C$

$\ln|y| = -2x^2 + C$

exponentiate $y = Ke^{-2x^2}$

$y(0) = 3 \Rightarrow K = 3$

c) $y(1) = 3e^{-2}$

d) error is $y(1) - y_{10}$

2. A detective finds a murder victim in a room with constant temperature 21°C. At 5:00am, the body's temperature was 34.2°C. One hour later, it was 31.8°C. Normal body temperature is 37°C. Assume the body's temperature, $B(t)$, follows Newton's Law of Cooling.
(a) Set up and solve the differential equation for $B(t)$.
(b) Estimate the time of the murder.

Solution: **Answers:** (a) $B(t) = \boxed{21 + 13.2e^{-0.2007t}}$ (b) murder occured at $\boxed{4:00}$

a) $$\frac{dB}{dt} = -k(B-21)$$

separate $$\frac{dB}{B-21} = -k\,dt$$

integrate $$\int \frac{dB}{B-21} = \int -k\,dt + C$$

$$\ln|B-21| = -kt + C$$

exponentiate $$B - 21 = Ke^{-kt}$$

or $$B(t) = 21 + Ke^{-kt}$$
(general solution)

let 5:00 be time 0

then $B(0) = 21 + K = 34.2$

so $K = 13.2$

then $B(t) = 21 + 13.2e^{-kt}$

but $B(1) = 21 + 13.2e^{-k} = 31.8$

so $13.2e^{-k} = 10.8$

thus $k = -\ln(10.8/13.2)$
$= 0.2007$

b) the murder occured at time when body was last at 37°C

so $37 = 21 + 13.2e^{-0.2007t}$

or $13.2e^{-0.2007t} = 16$

so $$t = \frac{\ln(16/13.2)}{-0.2007}$$
$= -0.96$ hrs

ie approx 1 hour before body discovered

3. A reservoir contains 500 litres of pure water. Brine (salty water) that contains 0.1 kg/L of salt is added at a rate of 2 L/min. Brine from a second source with 0.05 kg/L of salt is added at a rate of 3 L/min. Assume that the reservoir is instantaneously well-mixed. The reservoir is drained at a rate of 5 L/min. Let $Q(t)$ be the amount of salt (in kg) at time t (in min).

(a) Set up and solve the differential equation for $Q(t)$.

(b) How much salt is there in the reservoir after 4 hours?

Solution: **Answers:** (a) $Q(t) = \boxed{35(1-e^{-0.01t})}$ (b) $\boxed{31.82}$ kg

a) $\frac{dQ}{dt}$ = rate in − rate out

$= (0.1)(2) + (0.05)(3) - \frac{Q}{500}(5)$

$= 0.35 - \frac{Q}{100}$

$= -0.01(Q - 35)$

separate $\frac{dQ}{Q-35} = -0.01\,dt$

integrate $\int \frac{dQ}{Q-35} = \int -0.01\,dt + C$

So $\ln|Q-35| = -0.01t + C$

exponentiate $Q(t) = 35 + Ke^{-0.01t}$

but $Q(0) = 35 + K = 0$

$K = -35$

b) 4 hours is 240 minutes

So $Q(240)$

$= 35(1 - e^{-0.01(240)})$

$= 35(1 - e^{-2.4})$

$= 35(1 - 0.0907)$

4. A conservation group estimates that a wildlife preserve can sustain a herd of 2000 gnus. They also know that the relative growth rate of gnus would be 0.125 (per year) in an unconstrained environment. A herd of 800 gnus is placed in the preserve.
(a) Give the formula for the number $P(t)$ of gnus after t years, assuming that the population follows the Logistic Model.
(b) How long will it take for the population to grow to 1000?

Solution: **Answers:** (a) $P(t) = \boxed{\dfrac{4000}{2+3e^{-0.125t}}}$ (b) $\boxed{3.24}$ years

a) told that $M = 2000$, $k = 0.125$; $P_0 = 800$

so $$P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-kt}} = \frac{(2000)(800)}{800 + 1200e^{-0.125t}}$$

b) $$P(t) = 1000 \Rightarrow 1000 = \frac{4000}{2+3e^{-0.125t}}$$

or $$2 + 3e^{-0.125t} = 4$$

$$3e^{-0.125t} = 2$$

$$t = \frac{\ln(2/3)}{-0.125}$$

5. Do the following sequences converge or diverge? If they converge, give the limit.

(a) $a_n = \dfrac{(-1)^n\, 2n}{n^2+3n}$

(b) $a_n = \sin(n+1)$

(c) $a_n = \dfrac{5n^2+6n+3}{3n^2+7}$

(d) $a_n = \dfrac{3^{n+1}}{2^{n+3}}$

Solution: **Answers:** (a) $\boxed{\text{converges to } 0}$ (b) $\boxed{\text{diverges}}$

(c) $\boxed{\text{converges to } 5/3}$ (d) $\boxed{\text{diverges}}$

a) $\displaystyle \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n 2n}{n^2+3n} = \lim_{n\to\infty} \frac{(-1)^n 2/n}{1+3/n} = 0$

b) $\displaystyle \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin(n+1)$ does not exist

c) $\displaystyle \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5n^2+6n+3}{3n^2+7} = \lim_{n\to\infty} \frac{5+6/n+3/n^2}{3+7/n^2} = 5/3$

d) $\displaystyle \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3^{n+1}}{2^{n+3}} = \lim_{n\to\infty} \frac{3}{8}\left(\frac{3}{2}\right)^n = \infty$