Solutions

MAT 1322A – Assignment (due on Monday January 24, 10am)

Last Name (please print): First Name:

1. Use the Comparison Test to determine if the integral converges or diverges. Justify your answer in a clear manner.

(i) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}+x^4}\,dx\,,$ (ii) $\displaystyle\int_0^1 \frac{e^x}{x^3}\,dx\,.$

Solution:

i) as $x \to \infty$, $\dfrac{1}{\sqrt{x}+x^4}$ will behave like $\dfrac{1}{x^4}$

actually, for $x \geq 1$ $x^4 \leq \sqrt{x} + x^4$

and so $\dfrac{1}{\sqrt{x}+x^4} \leq \dfrac{1}{x^4}$

thus $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}+x^4}\,dx \leq \int_1^\infty \frac{1}{x^4}\,dx = 1/3$ (converges $p=4>1$)

$\therefore$ integral is convergent

ii) for all x in $[0,1]$, $1 \leq e^x \leq e$,

so we have that $\dfrac{1}{x^3} \leq \dfrac{e^x}{x^3}$

and thus $\displaystyle\int_0^1 \frac{1}{x^3}\,dx \leq \int_0^1 \frac{e^x}{x^3}\,dx$

but $\displaystyle\int_0^1 \frac{1}{x^3}\,dx$ is known to diverge

and $\therefore$ $\displaystyle\int_0^1 \frac{e^x}{x^3}\,dx$ diverges

2. Let R be the region bounded by the curves $y = 1/x$, $y = -1/x$ and the vertical lines $x = 1$ and $x = 2$. Denote by S the solid obtained by rotating R about the axis $y = 2$.

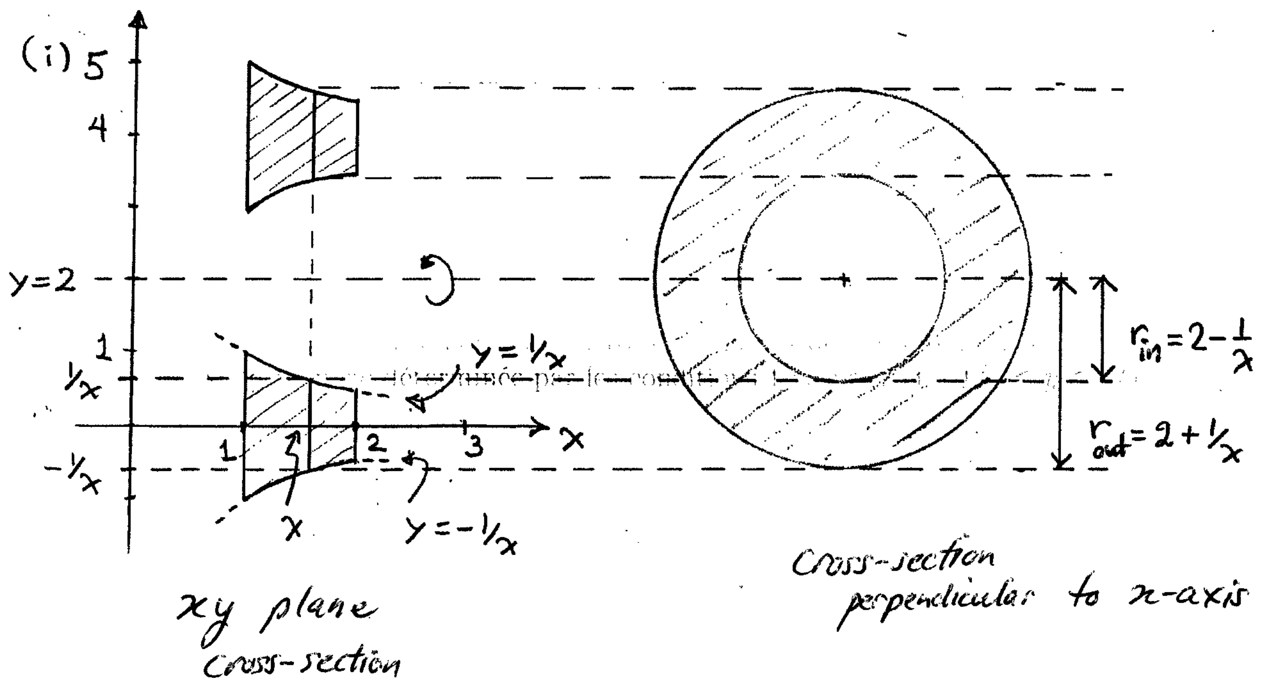
(i) Sketch the cross-section of the solid with the xy-plane. Also sketch the cross-section S_x of the solid S with the plane through x ($1 \le x \le 2$) and perpendicular to the x-axis.

(ii) What is the area $A(x)$ of the cross-section S_x?

(iii) Using the result from (ii) calculate the volume of S.

Solution:

(i)



ii) $A(x) = \pi\,(r_{out}^2 - r_{in}^2) = \pi\left[(2+1/x)^2 - (2-1/x)^2 \right]$

$= \pi\left[4 + 4/x + 1/x^2 - (4 - 4/x + 1/x^2)\right] = \boxed{\frac{8}{x}\pi}$

iii) the volume is

$$V = \int_1^2 A(x)\,dx$$

$$= \pi \int_1^2 \frac{8}{x}\,dx$$

$$= 8\pi \ln x \Big|_1^2$$

$$= \boxed{8\pi \ln 2}$$

$$\simeq \boxed{17.42}$$

3. Let R be the region in the xy-plane determined by the conditions $2 \le x \le 3$ and $1/x \le y \le 2/x$. Let S be the solid that is obtained by rotating R about the vertical line $x = 1$. Follow the steps below to calculate the volume of S by the method of cylindrical shells.

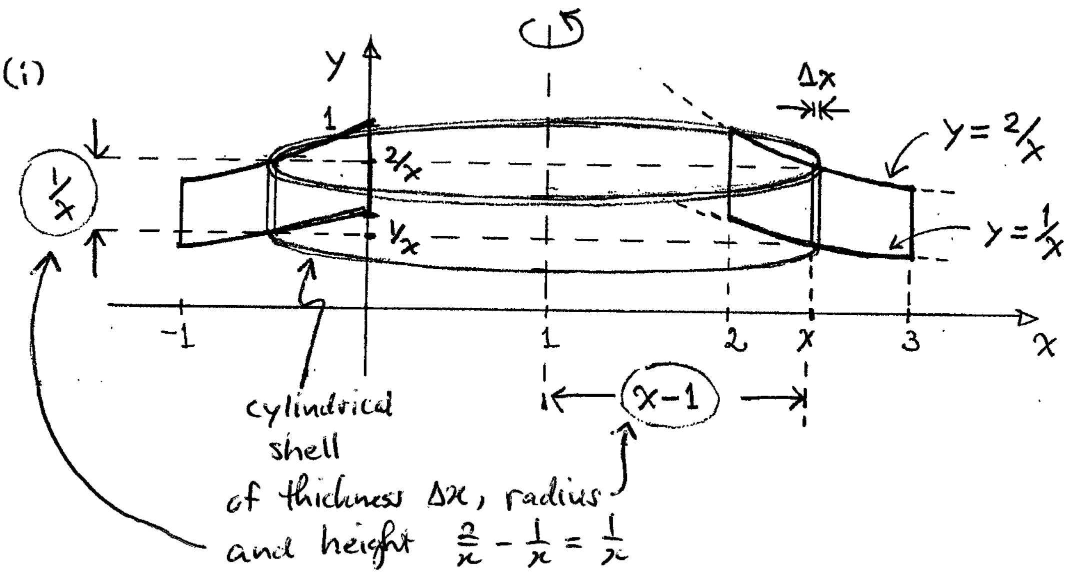
(i) Sketch the cross-section of the solid with the xy-plane. Also sketch a typical cylindrical shell of inner radius x and outer radius $x + \Delta x$ for $2 \le x \le 3$ and Δx small. Include dimensions in your sketches.

(ii) Give an approximation for the volume of the cylindrical shell in (i) as a function of Δx and x.

(iii) Using the result from (ii), calculate the volume of S.

Solution:

(i)



ii) volume of shell $\approx$ circumference $\times$ height $\times$ thickness

$$= \boxed{2\pi (x-1)\frac{1}{x}\Delta x}$$

iii) volume

$$V = \int_2^3 2\pi (x-1)\frac{1}{x}\,dx$$

$$= 2\pi \int_2^3 \left(1 - \frac{1}{x}\right)dx$$

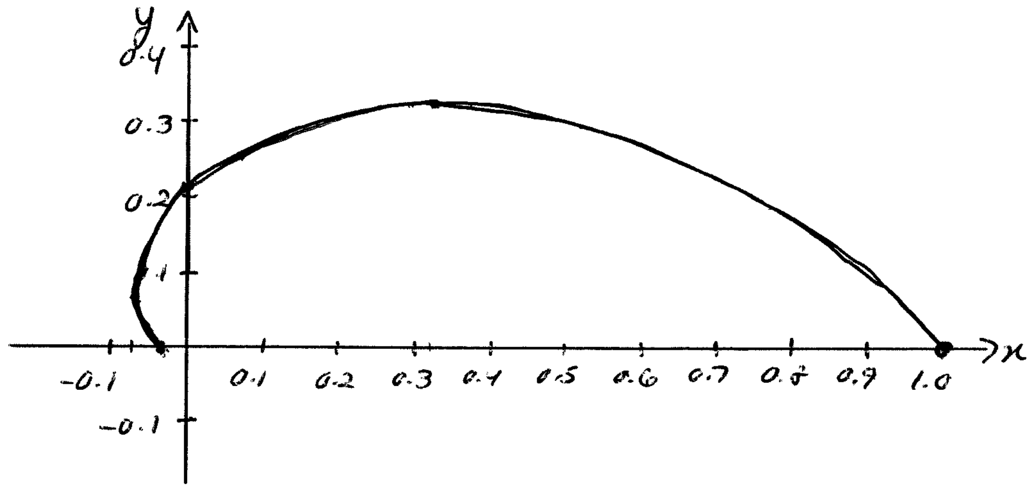
$$= 2\pi \left(x - \ln x \Big|_2^3\right)$$

$$= 2\pi \left[3 - \ln 3 - (2 - \ln 2)\right]$$

$$= \boxed{2\pi(1 - \ln 3 + \ln 2)} \simeq \boxed{3.7356}$$

4. Sketch the arc $x = e^{-t}\cos(t)$, $y = e^{-t}\sin(t)$, $0 \le t \le \pi$ and calculate its exact length. Note: After a simplification the integral is not very complicated.

Solution:



the length is $\quad L = \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

but $\quad \frac{dx}{dt} = -e^{-t}\cos t - e^{-t}\sin t$

and $\quad \frac{dy}{dt} = -e^{-t}\sin t + e^{-t}\cos t$

so $$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{-2t}\cos^2 t + 2e^{-2t}\cos t\sin t + e^{-2t}\sin^2 t$$
$$+ e^{-2t}\sin^2 t - 2e^{-2t}\cos t \sin t + e^{-2t}\sin^2 t$$
$$= 2e^{-2t}$$

so $$L = \sqrt{2}\int_0^{\pi} e^{-t}\,dt = -\sqrt{2}\,e^{-t}\Big|_0^{\pi}$$
$$= \boxed{\sqrt{2}\,(1 - e^{-\pi})}$$
$$\simeq \boxed{1.3531}$$