These are only some comments. You must come to the lectures to see/get the complete notes, solutions, graphs, how we apply formulae, etc. It may contain errors :).

Comments on lecture 1

1.1 Some ways to represent functions

Before we start we recall that **R** stands for the set of all real numbers, the symbol $\in$ means 'belongs', e.g., $\frac{-4}{2010} \in \mathbf{R}$.

Definition 0.0.1. *Let A, B be 2 sets (usually subsets of* **R** *- the set of real numbers). A function $f : A \mapsto B$ is a rule that assigns to each element $x \in A$ a unique element, $f(x)$, in B.*

• A is called **the domain** of the function f; the set of all possible values of $f(x)$ is called **the range** of the function.

Example 0.0.1. $f : \mathbf{R} \mapsto \mathbf{R}$, $f(x) = 2x + 1$ *for all* $x \in \mathbf{R}$. *What does it represent geometrically?*

TO VISUALIZE:

Definition 0.0.2. *THE GRAPH of a function $f : A \mapsto B$ is the set $\{(x, f(x)) | x \in A\}$*

Example 0.0.2. *Graph $f(x) = 2x + 1$ for all $x \in \mathbf{R}$. Note that $(0, 1)$ and $(1, 3)$ are in the graph, then just join them BY a line!*

Example 0.0.3. *Graph $f(x) = 2x^2$ for all $x \in \mathbf{R}$. Note that $(0, 0)$, $(1, 2)$ and $(-1, 2)$ are in the graph, then just join them by curve! What is the shape?*

The ways to represent a function are:
a) — verbally (using words)
b) — numerically (table of values)
c) — algebraically (using formulae)
d) — visually (using a graph)

You already encountered c), d). For a) think about $P : \mathbf{R} \mapsto \mathbf{R}$, $P(t)$ is the human population of Ottawa at time t. For b) think about a cheap example as follows:

x	1	2	3
$f(x)$	2	-1	-1

Do: 30,32/page 23

FOR FUTURE LECTUREs (WE NEED it FOR WHAT IS CALLED CALCULUS):

Definition 0.0.3. *The difference quotient is given by $\frac{f(a+h)-f(a)}{h}$, where $h \neq 0$ (and of course f and a are given).*

Do: 26,28/page 23

Natural Question: Is every curve (in the xy-plane) the graph of a function?

The answer is :

The Vertical Line Test: A curve in the xy-plane is the graph of a function (of x) if and only if ($\Leftrightarrow$) no vertical line cuts the curve more than once.

Example 0.0.4. *Think about the circle, parabola and other graphs...*

Definition 0.0.4. *A PIECEWISE DEFINED FUNCTION is a function defined by different formulae in different parts of the domain.*

Example 0.0.5. $f(x) = \begin{cases} x + \frac{7}{2} & \textit{if } x \geq -1\,, \\ -2 & x < -1\,. \end{cases}$

Graph it!

Important Example (The absolute value)

If $a \in \mathbf{R}$, then $|a| = \begin{cases} a & \text{if } a \geq 0\,, \\ -a & \text{if } a < 0\,. \end{cases}$ is called the absolute value of a. Define now the function $f : \mathbf{R} \mapsto \mathbf{R}_+$, $f(x) = |x|$. What is the range? Graph it!

Do: 42/page 23 and READ example 9/page 19.

Symmetry

Definition 0.0.5. *a) A function f is called EVEN if $f(x) = f(-x)$ for all x in its domain.*

b) A function is called ODD if $f(x) = -f(-x)$ for all x in its domain.

Example 0.0.6. $f(x) = x^2$, $f(x) = 2x^3$

Significance: a) The graph of an even function is symmetric with respect to the y-axis;

b) The graph of an odd function is symmetric about the origin $(0, 0)$.

Do: 72,70 /page 25

DECREASING AND INCREASING FUNCTIONS

It is what you have expected:

Definition 0.0.6. $f : I \mapsto \mathbf{R}$ *is called increasing on I if $f(x_1) < f(x_2)$ provided $x_1 < x_2$ in I.*

$f : I \mapsto \mathbf{R}$ *is called decreasing on I if $f(x_1) > f(x_2)$ provided $x_1 < x_2$ in I.*

Question: Is $f(x) = x^2$ increasing?

1.2 Important functions

a) **Linear functions:** $f : \mathbf{R} \mapsto \mathbf{R}$, $f(x) = mx + b$, where m, b are real numbers. Slope is m; b is the y-intercept.

Do: 16/page 36

b) **Polynomials:** $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0$; degree n if $a_n \neq 0$; $a_n, a_{n-1}, \ldots, a_1, a_0$ are called the coefficients of the polynomial.

Example 0.0.7. *1) A cubic function is just a polynomial of degree 3: $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, $a_3 \neq 0$*

2) A quadratic is just a polynomial of degree 2: $P(x) = a_2x^2 + a_1x + a_0$, $a_2 \neq 0$. Graph it for $\pm$ leading coefficient!

c) **Power function:** $f(x) = x^a$, where a is a real number.

Important:

— If $a = -1$, $f(x) = x^{-1} = \frac{1}{x}$, $x \neq$

— If $a = \frac{1}{n}$, n positive integer, $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$

d) **Rational function:** $f(x) = \frac{P(x)}{Q(x)}$, where P, Q are polynomials.

e) **Algebraic function:** One obtained from polynomials using $+, -, \times, \div, \sqrt[n]{}$.

Definition 0.0.7. $f(x) = \sqrt[7]{x - \frac{x}{2x+5}}$

f) **Trigonometric functions:** Sine, cosine, tangent = sin over cos etc...

g) **Exponential function:** $f(x) = a^x$, where a — the base — is a positive number; graph $f(x) = 2^x$, $g(x) = (\frac{1}{5})^x$.

h) **Logarithmic function:** $f(x) = \log_a x$, the base a is a positive number. It is the *inverse* of the EXP function.

NOTE THAT: $\log_a x = y$ iff $x = a^y$.

Do: 2/page 35

Comments on lecture 2

1.3 Getting new functions from old ones

Recall (otherwise read) the TRANSLATIONS (TO THE LEFT, RIGHT, UPWARD, DOWNWARD) + STRETCHING and REFLECTING

MORE IMPORTANT:

Combinations Given f and g one may define: $f + g$, $f - g$, fg, $\frac{f}{g}$ as follows

$(f + g)(x) = f(x) + g(x)$, $(f - g)(x) = f(x) - g(x)$, $(fg)(x) = f(x)g(x)$, $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$. The domain of the last one is the set of all x such that $g(x) \neq 0$ and $f(x)$ is defined. Find the domains of the others!

Do: 30/page 44

COMPOSITION Given f and g one defines their composition $f \circ g$ as follows $(f \circ g)(x) = f(g(x))$; what about the domain?

Do: 48,34,46/pages 44;45

1.5 EXPONENTIAL FUNCTIONS

Definition 0.0.8. $f(x) = a^x$, *where* $a > 0$

WHAT IS IT?

— If $x = n > 0$ then $a^x = a^n = a \times a \times \dots a$
— If $x = 0$ then $a^0 = 1$
— If $x = -n < 0$ then $a^x = a^{-n} = \frac{1}{a^n} = (\frac{1}{a})^n$
— If $x = \frac{p}{q}$ is rational then $a^x = a^{\frac{p}{q}} = \sqrt[q]{a^p}$ etc
— If x is irrational...

Laws of exponents:
— $a^{x+y} = a^x a^y$
— $a^{x-y} = \frac{a^x}{a^y}$
— $(a^x)^y = a^{xy}$
— $(ab)^x = a^x b^x$

The Number e

e is the positive number a such that the slope of the tangent line (=?) at $(0, 1)$ to $f(x) = a^x$ IS 1. For the moment $e \approx 2.71828$

Do: 20,30/pages 59

Comments on lecture 3

1.6 Inverse functions and LOGs

Definition 0.0.9. *A function $f : A \mapsto B$ is called one-to-one if it never takes on the same value: $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.*

Horizontal Line test says: A function is 1-1 (one-to-one) if and only if no horizontal line cuts its graph more than once.

Equivalent condition to get 1-1:

$f(x_1 = f(x_2)$ implies that $x_1 = x_2$.

Example 0.0.8. *Which onea are 1-1?*

$f(x) = x^3$, $g(x) = x^2$, $h(x) = x^2 - 2x$, $k(x) = \frac{1}{x}$.

Definition 0.0.10. *Let $f : A \mapsto B$ be a 1-1 function, assume B IS the range of f. Then its inverse function $f^{-1} : B \mapsto A$ is defined by the rule: $f^{-1}(f(x)) = x$.*

In other words $f^{-1}(y) = x$ iff $y = f(x)$. Note that Domain of f is the Range of f^{-1}, while the range of f is the domain of f^{-1}.

Algorithm (for finding the inverse)

— Set $y = f(x)$

— solve for x (in terms of y)

— interchange $x \leftrightarrow y$ and GET $f^{-1}(x)$.

D0: 22/page 70.

Keep in mind that: $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

This tells you that: the **GRAPH** of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

PARTICULAR CASE: Consider $f(x) = a^x$ for some $a > 0$, i.e., the exponential function. From its graph you may see that it is 1-1. So it has an inverse! It is denoted by $\log_a : (0, \infty) \mapsto \mathbf{R}$. Note that $f(x) = a^x = y \Leftrightarrow \log_a y = x$, $\log_a a^x = x$ and $a^{\log_a x} = x$.

NOTE: If $a = e$, than $\log_e := \ln$.

Do: 26/70.

Laws of LOGs: $\ln(xy) = \ln(x) + \ln(y)$, $\ln(\frac{x}{y}) = \ln(x) + \ln(y)$, $\ln(x^r) = r\ln(x)$.

Do: 51b, 52a, 49b/

Do (if time) 16, 15/70.

App. C: Trigonometry

Recall $\sin, \cos, \tan$ etc, their graphs, ranges, domains, periods.

Angles: Link between degrees and radians

csc, sec etc

Identities.

Comments on lecture 4

2.1 The tangent and velocity questions

I The tangent question/problem

— tangent line = line that touches/kisses the graph only in 1 point!

— problem: find the equation of the tangent line; so we need the slope of the tangent line!

— idea: (i) fix $P(x_0, y_0)$ on the graph, pick $Q(x, y)$ on the graph of the given function $f(x)$;

(ii) join P and Q; compute the slope of the secant line PQ as follows: $\frac{y-y_0}{x-x_0}$;

(iii) when Q approaches P, the slope of the tangent line is the limit of the slopes of the secant lines QP: $m = \lim\limits_{Q \to P} \frac{y-y_0}{x-x_0}$

II The velocity question/problem

— Suppose that we throw our book from a tower, and the distance (after t seconds) is given by $d(t) = 4.98t^2$.

— Question: What is the velocity after 5 seconds?

— We can attack the problem (and finding an estimate) by computing: average velocity $= \frac{change \quad in \quad position}{change \quad in \quad time} = \frac{d(5.1)-d(5)}{5.1-5} = \ldots.$

— if we want a better estimate of what is going on at 5 (from the point of view of velocity) we might compute $\frac{d(5.01)-d(5)}{5.1-5}$, or even $\frac{d(5.001)-d(5)}{5.1-5}$, ...

Do 7 on page 94

2.2 The limit of a function

Definition 0.0.11. *We write $\lim_{x \to a} f(x) = L$, and say the limit of $f(x)$ when x approaches a is L, if we can make the values of $f(x)$ arbitrary close to L by taking x sufficiently close to a (on either side), BUT not a.*

Example 0.0.9. $\lim_{x \to 1} \frac{x-1}{x^2-1} = \ldots$

Do 21/103;

Similarly, one may define $\lim_{x \to a^-} f(x) = L$, and $\lim_{x \to a^+} f(x) = L$.

Example 0.0.10. *If $f(x) = 1 + x$ if $x < -1$, $f(x) = x^2$ if $-1 \leq x < 1$, and $f(x) = 2 - x$ if $x \geq 1$, find $\lim_{x \to -1^-} f(x)$, $\lim_{x \to -1^+} f(x)$, $\lim_{x \to 1^-} f(x)$, $\lim_{x \to 1^+} f(x)$.*

Do 6 on 102 and 23 on 103.

Note that $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

2.3 The limit laws

Memorize all of them. Do 10-24/111.

2.4 Continuity

Definition 0.0.12. *A function g is called continuous at a if $\lim_{x \to a} g(x) = g(a)$.*

So g is defined in a and the value of the limit is exactly $g(a)$. A function is continuous if it is continuous at every point in the domain.

If f, g are continuous and c is a number, then $f \pm g$, cf, fg, f/g are continuous (for the last one as long as $g(a) \neq 0$). So polynomials, and rational functions (pay attention at the domain!) are continuous! Moreover: root functions, trig functions, exponential and logarithmic functions are continuous.

Theorem 0.0.1. $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$ *if f is continuous at $b = \lim_{x \to a} g(x)$.*

Theorem 0.0.2. *If f and g are continuous, then $f \circ g$ is continuous.*

Do 16 on 122, 36 on 123.

Theorem 0.0.3. *Assume that $f : [a, b] \to R$ is continuous, and let N be a number between $f(a)$ and $f(b)$ (here we assume $f(a) \neq f(b)$). Then there is a number c between a and b such that $f(c) = N$.*

DO 42, 44 on 123.

2.5 Limits dealing with ∞

I INFINITE LIMITS

Definition 0.0.13. *We write* $\lim_{x \to a} f(x) = \infty$*, and say the limit of* $f(x)$ *when* x *approaches* a *is INFINITY, if we can make the values of* $f(x)$ *arbitrarily large positive by taking* x *sufficiently close to* a *(on either side), BUT not* a*.*

Think of $\lim_{x \to 0} \frac{1}{x^4}$.

One may define in a similar fashion $\lim_{x \to a} f(x) = -\infty$, AND: $\lim_{x \to a^-} f(x) = -\infty$, $\lim_{x \to a^+} f(x) = -\infty$, $\lim_{x \to a^-} f(x) = \infty$, $\lim_{x \to a^+} f(x) = \infty$.

If at least one of the above 6 limits holds, then we say (by definition) that the line $x = a$ is a vertical asymptote!

II LIMITS at INFINITY

Definition 0.0.14. *We write* $\lim_{x \to \infty} f(x) = L$*, and say the limit of* $f(x)$*, when* x *approaches INFINITY, is* L *if we can make the values of* $f(x)$ *arbitrarily close to* L *by taking* x *sufficiently LARGE positive.*

One may define in a similar fashion $\lim_{x \to -\infty} f(x) = L$.

If at least one of the above 2 limits holds then we say that $y = L$ is a horizontal asymptote of f.

Try any from the range: 16-36 on page 133.

Comments on lecture 5

2.6 Derivatives and rates of change

Recall that one problem we discussed is the tangent problem. Given a function f, and a point $P(a, f(a))$ on the graph of f, we want to find the equation of the tangent line to the graph of f at P. The summary of 2.1 gives us the following:

Definition 0.0.15. *The tangent line to* $f(x)$ *at* $P(a, f(a))$ *is the line passing through* P *and has the slope* $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$*.*

Thus now you do have all ingredients to find the equation of a tangent line!

Do: 6/142.

A different look: as $x \to a$, note that $h = x - a \to 0$, so the formula for the slope can be written as follows:

$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$.

Definition 0.0.16. *The derivative of* f *at number* a *is denoted by* $f'(a)$ *and is given by* $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$*, if the limit exists.*

Of course $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$.

Moreover, the tangent line to the graph of f at the point $(a, f(a))$ is just the line (passing through $(a, f(a))$) whose slope is $f'(a)$.

If the motion of an object is described by a function f, then the (instantaneous) velocity is given by f'. The usual

average velocity is $\frac{change \quad in \quad position}{change \quad in \quad time}$. In other words the average velocity between a and $a + h$ is given by: $\frac{f(a+h)-f(a)}{h}$.

Do 16 a,b; 24; 18; 20; 26a; 34, 36, 38/143

Sol: a), b): (ii) $\frac{s(4)-s(3.5)}{4-3.5} = \frac{(16-32+18)-(3.5^2-8\times3.5+18)}{0.5} = -0.5$. Now, $\lim_{t\to4}\frac{f(t)-f(4)}{t-4} = \cdots = \lim_{t\to4} t+4-8 = 0$. Is (a) close enough for you to (b)?

2.7 Derivative as a Function

Definition 0.0.17. *The derivative of a given function f is a new function f' given by $f'(x) = \lim_{h\to0}\frac{f(x+h)-f(a)}{h}$.*

The domains of f and f' might be different!
Do 24/156.
Other notations: If $y = f(x)$ is given, then other notations for the derivative are: $f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{df(x)}{dx} = Df(x)$.

Definition 0.0.18. *We call a function f differentiable at a if $f'(a)$ exists. If f is differentiable at every point in an interval, then we call f differentiable on that interval.*

Example 0.0.11. *Is $f(x) = |x|$ differentiable?*

Theorem 0.0.4. *If f is differentiable at a, then f is continuous at a.*

Question: when is a function NOT differentiable?
Answers: corners, discontinuities, and vertical tangent lines!
Higher Derivatives
By definition we call $(f')'$ the second derivative of f, and we denote it by f''. The third derivative is $(f'')'$ and is denoted by f'''. As an application, recall that acceleration is in fact the derivative of velocity, so it is the second derivative of the position function!
Do 3/155; 26/155

Comments on lecture 6

2.8 What does the derivative says about the function?

Looking at a graph at the blackboard (and believing Chapter 4...) one has

Theorem 0.0.5. *If $f'(x) > 0$ on an interval, then f is increasing on that interval. If $f'(x) < 0$ on an interval, then f is decreasing on that interval.*

Definition 0.0.19. *f has a local maximum at a if for x near a one has that $f(a) \geq f(x)$.*
f has a local minimum at a if for x near a one has that $f(a) \leq f(x)$.

Do: 25/163
What does f'' say about f?
Recall that $f'' = (f')'$. If $f'' > 0$, then f' is increasing, and draw a graph now...
Recall that $f'' = (f')'$. If $f'' < 0$, then f' is decreasing, and draw a graph now...

Theorem 0.0.6. *If $f''(x) > 0$ on an interval, then f is concave upward on that interval. If $f''(x) < 0$ on an interval, then f is concave downward on that interval.*

Definition 0.0.20. *An inflection point is just a point where a curve changes its concavity.*

Do: 22/163, 28/163.

3.1 Derivatives of polynomoals and exponentials

Using the definition we derive the following rules:

Theorem 0.0.7. *The derivative of a constant function $f(x) = c$ is given by $(c)' = 0$.*

The power rule:

Theorem 0.0.8. *The derivative of a power function* $f(x) = x^n$ *is given by* $(x^n)' = nx^{n-1}$. *Here* n *is a real number.*

New Rules:
Constant Multiple rule:

Theorem 0.0.9. $(cf(x))' = cf'(x)$.

Sum and Difference rule:

Theorem 0.0.10. $(f(x) \pm g(x))' = f'(x) \pm g'(x)$.

Thus, from now on you may compute the derivative of any polynomial!
Try as many as possible from:
18, 20, 24, 10/181; 28, 39, 32, 29/181;
Recall the definition on number e and note that $\lim\limits_{h \to 0} \frac{e^h - 1}{h} = 1$.
Thus $(e^x)' = e^x$.
Do: 47, 54/182

Comments on lecture 7

3.2 Product and quotient rules

Theorem 0.0.11. *If* f *and* g *are both differentiable, then* $f(x)g(x)$ *is differentiable. Moreover, one has that* $\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x)$.

Theorem 0.0.12. *If* f *and* g *are both differentiable, then* $\frac{f(x)}{g(x)}$ *is differentiable. Moreover, one has that* $\{\frac{f(x)}{g(x)}\}' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$.

DO:36a, 30 a/189; 24, 18, 34a, 40, 20, 4 on page 188.

3.3 Derivatives of TRIG functions

Based on $\lim\limits_{x \to 0} \frac{\sin(x)}{x} = 1$ and on $\lim\limits_{x \to 0} \frac{\cos(x) - 1}{x} = 0$, one gets:

Theorem 0.0.13. $(\sin x)' = \cos(x)$ *and* $(\cos x)' = -\sin(x)$.

There is a table on page 194...
$\{\tan(x)\}' = \sec^2(x)$, $\{\cot(x)\}' = -\csc^2(x)$, $\{\sec(x)\}' = \sec(x)\tan(x)$, $\{\csc(x)\}' = -\csc(x)\cot(x)$, where $\sec(x) = \frac{1}{\cos(x)}$ and $\csc(x) = \frac{1}{\sin(x)}$.
Do: 13, 23a, 10, 6, 27/195

Comments on lecture 8

3.4 Chain Rule

The only theorem in this section is the Chain Rule:

Theorem 0.0.14. *If* f *and* g *are differentiable and* $f(g(x)$ *is defined, then* $f(g(x)$ *is differentiable, and* $\{f(g(x))\}' = f'(g(x))g'(x)$.

Please memorize the Chain rule and Power rule; and memorize Chain rule and exponentials!
Do: 12, 32, 26, 44/205; do 46a/205, do 50/205, 52/205.

3.5 Implicit differentiation

When $y = f(x)$ is 'given', one may find $y' = f'(x)$. What if y is not given?! What can we do if y is given in a relation? (Say that it is impossible to solve for y.)

— differentiate both sides of the equation with respect to the variable x;

— solve for y' (i.e., isolate y').

Do: 10/214, 18, 17, 15, 13/214 and 23, 25/205.

Comments on lecture 9

3.6 Inverse Trig Functions and (of course) their derivatives

The function $f(x) = \sin(x)$, $f : [-\pi/2, \pi/2] \to [-1, 1]$ IS one-to-one! Draw the graph! So the inverse exists, and it is called the **arcsine** function, denoted by $\sin^{-1}$. In other words, $\sin^{-1}(x) = y$ if and only if $\sin(y) = x$. By the very definition of the inverse of a function, one gets that $\sin^{-1}(\sin(x)) = x$ and $\sin(\sin^{-1}(x)) = x$. Using Implicit Differentiation one can get that $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ for all $-1 < x < 1$.

The function $f(x) = \cos(x)$, $f : [0, \pi] \to [-1, 1]$ IS one-to-one! Draw the graph! So the inverse exists, and it is called the **inverse cosine** function, denoted by $\cos^{-1}$. In other words, $\cos^{-1}(x) = y$ if and only if $\cos(y) = x$. By the very definition of the inverse of a function, one gets that $\cos^{-1}(\cos(x)) = x$ and $\cos(\cos^{-1}(x)) = x$. Using Implicit Differentiation one can get that $\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$ for all $-1 < x < 1$.

The function $f(x) = \tan(x)$, $f : (-\pi/2, \pi/2) \to \mathbf{R}$ IS one-to-one! Draw the graph! So the inverse exists, and it is called the **inverse tangent** function, denoted by $\tan^{-1}$. In other words, $\tan^{-1}(x) = y$ if and only if $\tan(y) = x$. By the very definition of the inverse of a function, one gets that $\tan^{-1}(\tan(x)) = x$ and $\tan(\tan^{-1}(x)) = x$. Using Implicit Differentiation one can get that $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$.

Do (many) problems from page 220.

Comments on lecture 10

3.7 Derivatives of the LOG functions

Theorem 0.0.15. *(i)* $\{\log_a(x)\}' = \frac{1}{x\ln(a)}$;

(ii) $\{\ln(x)\}' = \frac{1}{x}$.

Let us do a proof: if $\log_a(x) = y$, then $a^y = x$; by implicit differentiation one gets $a^y \ln(a) y' = 1$. Thus $y' = \frac{1}{a^y \ln(a)} = \frac{1}{x\ln(a)}$. For (ii) use $\ln(e) = 1$.

Do: 14,4,6,28/page 226

New method: **logarithmic differentiation**.

— Take ln of both sides of $y = f(x)$, use laws;

— Use Implicit differentiation;

— Find y'.

Do: 34, 38/page 226.

3.9 Linear approximations and differentials

Story time...

Thus, it makes sense the following

Definition 0.0.21. $f(x) \approx f(a) + f'(a)(x - a)$ *is called the linear approximation (or tangent line approximation) of* f *at* a.

Definition 0.0.22. $L(x) = f(a) + f'(a)(x - a)$ *is called the linearization of* f *at* a.

Do 6,8/page 245, and then do 9/245!

Comments on lecture 11

Chapter 4: Applications of differentiation

4.1 Some related rates

Read examples from section 4.1; AND THE PLAN ON PAGE 258!!!
— read the problem (even few times);
— draw (if necessary);
— introduce notations;
— use rates;
— use equations;
— differentiate (using chain rule or other rules);
— get the unknown rate.
Do 6, 14/page 260.

4.2 Talk about maximum and minimum VALUES

Definition 0.0.23. *Let c be a number in the domain of a function $f : D \to \boldsymbol{R}$. Then $f(c)$ is called:*
— absolute maximum value of f on D if $f(c) \geq f(x)$ for all x in D;
— absolute minimum value of f on D if $f(c) \leq f(x)$ for all x in D.

Draw yourself a graph to get the idea!
You may call these values: GLOBAL MAXIMUM/MINIMUM, or EXTREME VALUES OF f.

Definition 0.0.24. *The number $f(c)$ is called:*
— local maximum value if $f(c) \geq f(x)$ when x is near c;
— local minimum value if $f(c) \leq f(x)$ when x is near c;

Example 0.0.12. sin*,* cos *have infinitely many extreme values.*
$f(x) = x^2$ has an absolute(local) minimum value, BUT there are no local(global) maximum values!
$g(x) = x^3$ has no local max or min!

We can we say?

Theorem 0.0.16. *Any continuous function $f : [a, b] \to \boldsymbol{R}$ attains an absolute maximum value and an absolute minimum value.*

Closed interval! Continuous!
What can we say about local max/min? Fermat's Theorem:

Theorem 0.0.17. *If f has a local maximum or minimum at c, and $f'(c)$ exists, then $f'(c) = 0$.*

The converse is not true! Think about x^3.

Definition 0.0.25. *A critical number of f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.*

The closed interval method:
To find the absolute maximum and minimum values of a function $f : [a, b] \to \mathbf{R}$ do:
— find the values of f at the critical numbers in (a, b);

— find $f(a)$ and $f(b)$;

— the largest of values from steps 1, 2 above is the absolute maximum; the smallest of values from steps 1, 2 above is the absolute minimum.

Do problems from 268-269.

Comments on lecture 12

4.3 Derivatives + Shapes of curves

We start by the MEAN VALUE THEOREM:

Theorem 0.0.18. *Suppose that f is differentiable on the closed bounded interval $[a, b]$. Then There is a number c between a and b such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.*

TRY yourself to give a picture for the proof of this fact!

DO exercises:64/282;63/282.

Next let us talk about increasing and decreasing functions (and think about the above theorem):

Theorem 0.0.19. *(i) If $f'(x) > 0$ on an interval, then f is increasing on this interval;*
(ii) If $f'(x) < 0$ on an interval, then f is decreasing on this interval;

Do Exercises: 13,15/280

The first derivative test is:

Theorem 0.0.20. *Suppose that c is a critical number of f.*
(i) if f' changes from $+$ to $-$ at c, then f has a local maximum at c;
(ii) if f' changes from $-$ to $+$ at c, then f has a local minimum at c;
(iii) if f' does not change sign at c, then f has no local maximum or minimum at c.

Definition 0.0.26. *A function f (or/and its graph) is called CONCAVE upward on an interval if f' is increasing on that interval. It is called CONCAVE downward if f' is decreasing.*

There is a **concavity test:**

Theorem 0.0.21. *(i) If the second derivative of f satisfies $f''(x) > 0$ for all x in an interval I, then the graph of f is concave upward on the interval I;*
(ii) If the second derivative of f satisfies $f''(x) < 0$ for all x in an interval I, then the graph of f is concave downward on the interval I.

Now think about **the second derivative test**:

Theorem 0.0.22. *Assume that f'' is continuous near the number c:*
(i) If $f'(c) = 0$ and if $f''(c) > 0$, then f has a local minimum at c;
(ii) If $f'(c) = 0$ and if $f''(c) < 0$, then f has a local maximum at c;

DO exercises: 17/280; 37/280, 41/280, 31/280.

Comments on lecture 13

4.5 Indeterminate forms and L'Hopital rule

A limit of the form $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ where $\lim\limits_{x\to a} f(x) = 0 = \lim\limits_{x\to a} g(x)$, is called an **indeterminate form of type $\frac{0}{0}$**. Think about $\lim\limits_{x\to 1} \frac{x^2-1}{x^2-x}$. Compute it!

Another example: do you recall that $\lim\limits_{x\to 0} \frac{\sin(x)}{x} = 1$? At the beginning is of type $\frac{0}{0}$.

A limit of the form $\lim_{x\to a} \frac{f(x)}{g(x)}$ where $\lim_{x\to a} f(x) = \pm\infty = \lim_{x\to a} g(x)$, is called an **indeterminate form of type** $\frac{\infty}{\infty}$. Think about $\lim_{x\to\infty} \frac{x^2-1}{x^2-x+1}$. Compute it!

Theorem 0.0.23. *L'Hopital rule: Assume f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Assume either*
$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$
or
$\lim_{x\to a} f(x) = \pm\infty = \lim_{x\to a} g(x)$.
Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ IF THE LIMIT ON THE RIGHT SIDE EXISTS (or is $\pm\infty$).

NOTE: The expression $x \to a$ could be replaced by any of the following expressions: $x \to a^+$, $x \to a^-$, $x \to \infty$, $x \to -\infty$.

Read all examples at pages 292-293.

Indeterminate form of type $0 \times \infty$ can be dealt with as follows:
— the product fg can be rewritten as a quotient $\frac{f}{1/g}$ or $\frac{g}{1/f}$.

Indeterminate form of type $\infty - \infty$ can be dealt with as follows:
— the difference $f(x)-g(x)$ can be rewritten as a quotient or a product, by FACTORING.

Let us talk about some

INDETERMINATE POWERS

General form: $\lim_{x\to a} f(x)^{g(x)}$.

Types: 0^0, ∞^0, 1^∞.

TREATMENT: either take the natural logarithm (ln), or write the power as an exponential $f(x)^{g(x)} = e^{\ln(f(x)^{g(x)})} = e^{g(x)\ln(f(x))}$ — by the properties of ln.

Now do problems from pages 296-297.

Comments on lecture 14

4.6 Optimization

Plan:

— read the exercise, read it carefully; get the point; understand what is unknown, what are the quantities, what are the conditions;

— draw a diagram when necessary;

— introduce notation(s); use symbols V, f, x, y etc; let us fix V in our discussion;

— write V in terms of other symbols (think about functions now) introduced above;

— if V is expressed as a function of several variables, use equations (obtained from the statement or whatever) to eliminate some of the variables, your goal is to write V as a function of only one variable; (you do Calculus I this semester and not Calculus III); suppose that you are able to write $V(x) = \ldots$; find now the domain;

— find the absolute maximum or minimum;

Do 3/305. Let x and y be positive numbers. We know (by reading the text) that $xy = 100$. We want to minimize the sum $S = x + y$. Note that S depends on 2 variables... so we are in trouble. But recall that $xy = 100$. Hence $y = \frac{100}{x}$. Now $S(x) = x + \frac{100}{x}$ is a function of only one variable! Its domain is $(0, \infty)$ (just read again the statement!). Compute $S'(x)$ and fill in the table for $x, S'(x), S(x)$. Did you get that $x = 10$ is a critical number? What is the absolute minimum? Don't forget: we must find x and y. So $y = \frac{100}{10} = \ldots$.

Do 5/305. Say that x is the length and that y is the width. Then from the statement one gets that $2x + 2y = 100$. We want to maximize area $A = xy$. We write $y = 50 - x$, and thus $A(x) = x(50 - x)$. The domain of A is $(0, 50)$. Compute $A'(x) = -2x + 50$ and get the critical number $x = 25$ that is in the domain of A. Fill in the table of $x, A'(x), A(x)$. Is 25 an absolute maximum or minimum? What is the width?

Do 9/306

Do 11/306

Do 15/306

D0 25/206

Comments on lecture 15

4.7 Newton's Method

Say that we want to find **a solution** of the equation $f(x) = 0$, and that f is a monster, or no formula (like the quadratic formula for polynomials of degree 2) is available. Can you solve $\sin(74x - e^{x-89}) + \ln(x^7 + 23x) + x^{78} = 0$? I can not!

Say that the root (solution) we are trying to get is r. The 1st step is to guess, approximate r (think about Intermediate value theorem). Say the first approximation is x_1.

Do a graph! Construct the tangent line to the graph of f at the point $(x_1, f(x_1))$, and look at the picture... Say the x-intercept is x_2. Since the tangent line is close (near $(x_1, f(x_1))$) to the graph of f, we believe that the x-intercept of the tangent line is close to the x-intercept of f. BUT, **can** we find x_2? The equation of the tangent line is $y - f(x_1) = f'(x_1)(x - x_1)$. Plug in the point $(x_2, 0)$, and get $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$. We say that x_2 is the second approximation of our r.

Why stop here? Why not continue? What you did for x_1, do for x_2, and get the 3rd approximation $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$.

And so on...

The general formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

When the method fails?

— when the initial approximation is not good; (so we could change the initial approximation, and start the process again);

— when $f'(x_n) = 0$ for some n (you cannot divide by 0);

— when there many other roots;

Try pages 315–316

4.8 Antiderivatives

Definition 0.0.27. *A function F is called an antiderivative of f on an interval J if $F'(x) = f(x)$ for all x in that J.*

Since the derivative of a constant is 0, we give the following

Theorem 0.0.24. *Let F be an antiderivative of f on an interval $J \subseteq \boldsymbol{R}$. Then the general antiderivative of f on that J is the bunch of functions $F(x) + c$, where $c \in \boldsymbol{R}$.*

There is a table on page 318. Memorize it!!! Do pages 321–322.

Comments on lecture 16

Chapter 5 INTEGRALS

5.1 Areas and distances

I. Let us talk about area problem: Everybody knows... the formula for the area of a triangle, square, rectangle, ... For some curved regions it is difficult to find the area. At the beginning we are going to estimate, approximate it! And see when the estimate is getting better.

Draw a picture (of a nice, in fact of its graph!) function. Assume f is positive. We shall use rectangles to approximate the area under that function, x-axis, and some given bounds!

Choose $f(x) = x^2$, between 0 and 1. Divide the interval $[0, 1]$ into 4 rectangles of equal size. There are 2 ways we can approximate: overestimating, underestimating!

Compute both estimates, and realize that the true value is somewhere in between...

THE big QUESTION: HOW CAN WE GET BETTER ESTIMATES?

ANSWER: INCREASE THE NUMBER OF SUBINTERVALS. WHAT IS HAPPENING WITH THE LENGTH OF THE SUBINTERVALS?

SO, let us generalize:

Consider the function $f : [a, b] \to \mathbf{R}$ be a continuous function. We want to find the area of the region under the graph of f, between a and b, and x-axis. Draw a picture! Divide your region into n strips s_1, s_2, ... , s_n of equal width. So the width is $\Delta x = \frac{b-a}{n}$.

The subintervals are $[x_0 = a, x_1 = a+\Delta x]$, $[x_1 = a+\Delta x, x_2 = a+2\Delta x]$, $[x_2 = a+2\Delta x, x_3 = a + 3\Delta x]$, etc.

An estimate is (using right end points of subintervals) $R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$.

Definition 0.0.28. *The area of the region under the graph of f, between a and b, and x-axis is* $A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \{f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x\}$.

The above limit exists BECAUSE f IS assumed to be continuous! Using left endpoints one gets that $A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \{f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x\}$. It can be proved that we can actually take any number x_i^* in the subinterval $[x_{i-1}, x_i]$. It is called (for each i) a sample point.

The sigma notation: $f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \Sigma_{i=1}^n f(x_i)\Delta x$. Can you write the left sum in sigma notation?

II. The distance problem

Recall (from your childhood) that distance = velocity $\times$ time.

Do 11/341 on distances.

Then about areas do 3,5/341.

5.2 Definite integral

Based on 5.1 we introduce

Definition 0.0.29. *If $f : [a, b] \to \boldsymbol{R}$ is a function, divide the domain (i.e., $[a, b]$) into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. Say that the endpoints of these subintervals are $x_0, x_1, \ldots, x_n$. Let $x_i^* \in [x_{i-1}, x_i]$ be any sample points. The definite integral of f from a to b is given and denoted by $\int_a^b f(x)dx = \lim_{n\to\infty} \Sigma_{i=1}^n f(x_i^*)\Delta x$ provided that the limit exists. If the limit exists, we call f integrable over $[a, b]$.*

— it is a number (when it exists);

— integral sign, integrand, lower and upper limits of integration;

— Riemann sum $\Sigma_{i=1}^n f(x_i^*)\Delta x$;

— when f is positive, we get back the area under f, the x-axis, a and b.
— if f is not necessarily positive, the integral is a net area, a difference of areas;
— the subintervals need not be of equal width;

Theorem 0.0.25. *If either f is continuous on the interval $[a,b]$, or f has a finite number of jump discontinuities, then f IS integrable over $[a,b]$.*

The midpoint rule: if the sample points are the midpoints of the subintervals (so $x_i = \frac{x_{i-1}+x_i}{2}$), then $\int_a^b f(x)dx = \lim_{n\to\infty} \Sigma_{i=1}^n f(\frac{x_{i-1}+x_i}{2})\Delta x$.

Properties of the integral:
$\int_a^b f(x)dx = -\int_b^a f(x)dx$;
$\int_a^a f(x)dx = 0$;
$\int_a^b cdx = c(b-a)$;
$\int_a^b \{f(x) \pm g(x)\}dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$;
$\int_a^b cf(x)dx = c\int_a^b f(x)dx$;
Very important:
$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$;
If $f(x) \geq 0$ for all x in $[a,b]$, then one has that $\int_a^b f(x)dx \geq 0$;
If $f(x) \geq g(x)$ for all x in $[a,b]$, then one has that $\int_a^b f(x)dx \geq \int_a^b g(x)dx$;
If $m \leq f(x) \leq M$ for all x in $[a,b]$, then one has that $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$;
DO: 5, 7, 9, 11, 19, 31, 37, 41, 43 on page 353

Comments on lecture 17

5.3 Evaluating some definite integrals

Newton and Leibniz:

Theorem 0.0.26. *If $f : [a,b] \to \boldsymbol{R}$ is a continuous function, then one has that $\int_a^b f(x)dx = F(b) - F(a)$, where F is ANY antiderivative of f, i.e., in other words $F' = f$.*

For a proof, see the book (page 356).
Do problems from page 363: try 1—30.
In part we talk about:
Indefinite integrals:
Terminology: $\int f(x)dx = F(x)$ stands for an antiderivative of f, and it is called an INDEFINITE integral. In other words $F'(x) = f(x)$.
This is a function, not a number!!! The evaluation theorem can be written as follows:
$\int_a^b f(x)dx = \int f(x)dx|_a^b$

Example 0.0.13. $\int \sin(x)dx = -\cos(x) + c$, *where c is a number.*

There is a table of integrals on page 358, read it/memerize it!!! If you know well the differentiation rules, it will be easier to understand and memorize these new rules!!!
Before we do exercises note the following NET CHANGE theorem:

Theorem 0.0.27. $\int_a^b F'(x)dx = F(b) - F(a)$

It is good to know it for applications, word problems etc.
Do:43— 48; 51; 53;

Comments on lecture 18

Let us talk about **the Fundamental Theorem of Calculus: 5.4**

Part I:

Theorem 0.0.28. *Consider a continuous function $f : [a, b] \to \boldsymbol{R}$, and then define a new function*

$g(x) = \int_a^x f(t)dt$, *where* $a \leq x \leq b$.

Then g is an antiderivative of f, i.e., $g'(x) = f(x)$ for all x in (a, b).

Part II:

Theorem 0.0.29. *Assume that $f : [a, b] \to \boldsymbol{R}$ is a continuous function.*

(a) IF $g(x) = \int_a^x f(t)dt$, Then $f(x) = g'(x)$;

(b) $\int_a^b f(x)dx = F(b) - F(a)$, if F is any antiderivative of f, i.e., $F' = f$.

DO: 3, 5, 7, 9, 13/373 and many other more from the same page!

Please memorize these theorems! You are going to use them in other math classes over the years! There are fundamental in Differential and Integral Calculus! **Realize** that there are exercises (see the textbook!) where we must use chain rule **and** the above theorem!!! See 15/373.

Comments on lecture 19

Let us talk about **SUBSTITUTION: 5.5**

Let us discuss a simple, but enlightening example: compute $\int 2010x\sqrt{x^2 + 2100}dx$.

What you noticed in the solution of the above exercise is this:

Theorem 0.0.30. *If $u = g(x)$ is a function whose derivative exists, and whose range is an interval J,*

and if f is a continuous function on J,

then $\int f(u)du = \int f(g(x))g'(x)dx$.

Example 0.0.14. *Compute $\int e^{-2010x}dx$.*

DO Problems from 1—40 on page 381.

Now there is a definite version of the above indefinite version of the SUB:

Theorem 0.0.31. *Assume that $g'(x)$ is continuous on the interval $[a, b]$, that f is continuous on the range of $u = g(x)$. Then one has that:*

$\int_{g(a)}^{g(b)} f(u)du = \int_a^b f(g(x))g'(x)dx$.

Do as many as possible from 41—57 on page 382!

Comments on lecture 20

Let us talk about **Integration by parts: 5.6**

We can deduce the counterpart of product rule:

Theorem 0.0.32. $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$.

Please read all examples (e.g., Example 4) done in 5.6; and realize that there is a definite integral version of it:

Theorem 0.0.33. $\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx$.

TRY: 1, 3, 7, 9, 11, 13, 17, 19, 21, 25, 27, 39, 41 on page 387!!!