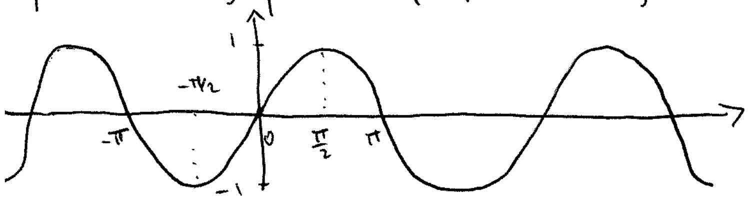
Lecture 2 — 19^{th} OF SEPTEMBER

HGN-302-

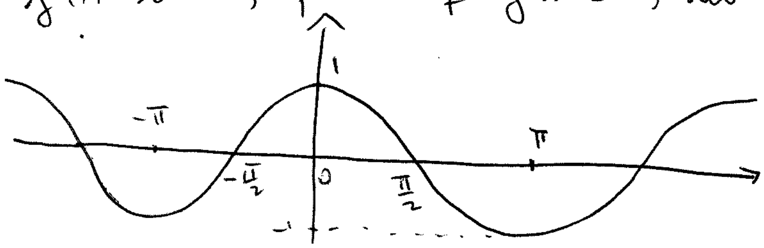
§ 1.2 TRIG. FUNCTIONS

$f(x) = \sin x$; period of f is 2π, domain $\mathbb{R}$, $-1 \le \sin x \le 1$



$\sin x = 0 \iff x = n\pi$; n in $\mathbb{Z}$

$g(x) = \cos x$; period of g is 2π ; domain $\mathbb{R}$, $-1 \le \cos x \le 1$

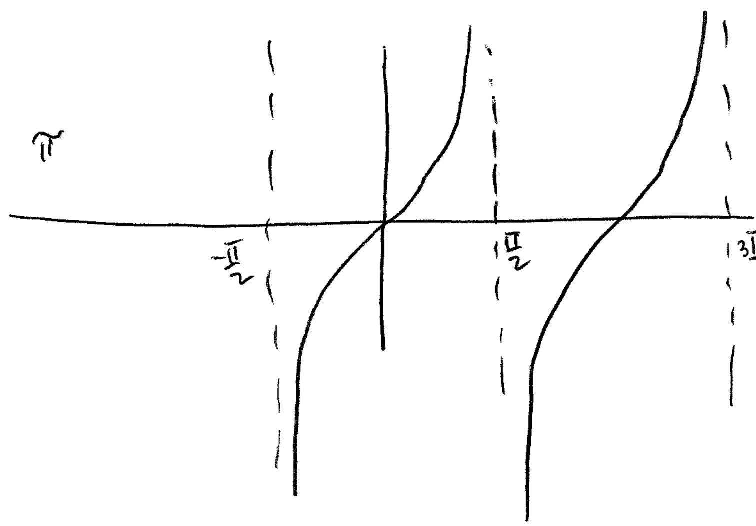


$\cos x = 0 \iff x = \frac{2n+1}{2} \cdot \pi$, $n \in \mathbb{Z}$

$h(x) = \tan x = \frac{\sin x}{\cos x}$; period of h is π

DOMAIN : $\mathbb{R} \setminus \{ \frac{2n+1}{2}\pi \mid n \in \mathbb{Z} \}$

Range : $(-\infty, \infty)$



§ 1.5 EXPONENTIAL FUNCTIONS

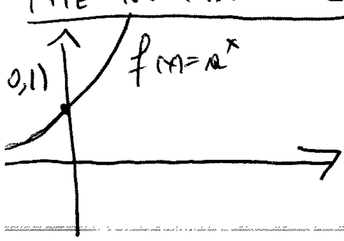
DEF: $f(x) = a^x$, $a > 0$

WHAT IS IT?

- $x = n > 0$, positive integer $\Rightarrow a^x = a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}$
- $x = 0 \Rightarrow a^x = a^0 = 1$
- $x = -n < 0 \Rightarrow a^x = a^{-n} = \left(\frac{1}{a}\right)^n = \underbrace{\frac{1}{a} \cdot \frac{1}{a} \cdot \ldots \cdot \frac{1}{a}}_{n \text{ times}}$
- $x = \frac{p}{q}$ rational $\Rightarrow a^x = a^{p/q} = \sqrt[q]{a^p}$
- $x =$ irrational ... STORY ...

LAWS OF EXPONENTS: $a^x \cdot a^y = a^{x+y}$; $\frac{a^x}{a^y} = a^{x-y}$; $(a^x)^y = a^{xy}$; $(ab)^x = a^x b^x$

THE NUMBER e: Consider $f(x) = a^x$ and its graph:



e is the positive # a s.t. the slope of the tangent line at $(0,1)$ to $f(x) = a^x$ is exactly 1

FOR the moment: $e \approx 2.71828$

ALGORITHM (for finding the inverse)

1) Set $y = f(x)$ 2) SOLVE FOR x (in terms of y)

3) Interchange $x \longleftrightarrow y$ and GET $f^{-1}(x)$.

DO:

$\boxed{22/70}$ $f(x) = \dfrac{4x-1}{2x+3}$. SOL: $\dfrac{4x-1}{2x+3} = y \Rightarrow 4x - 1 = 2xy + 3y$

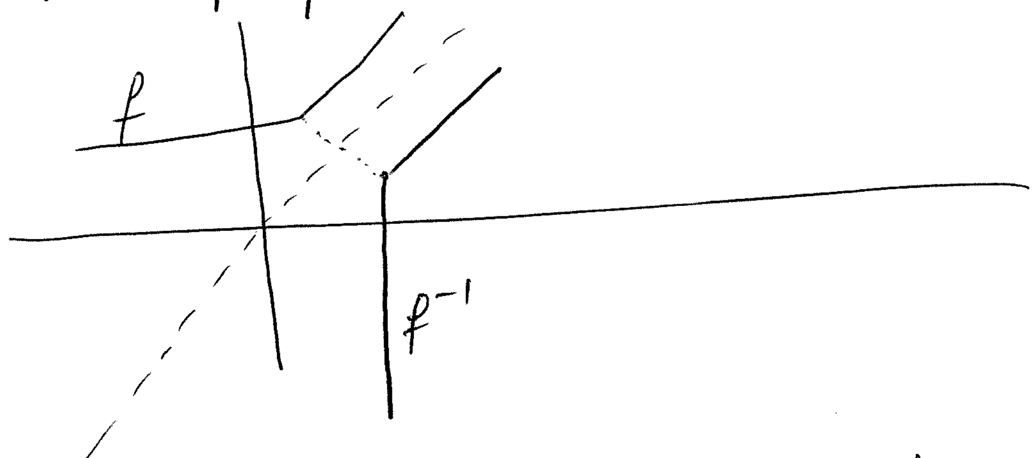
$\Rightarrow x(4-2y) = 1+3y \Rightarrow x = \dfrac{1+3y}{4-2y} \Rightarrow f^{-1}(x) = \dfrac{1+3x}{4-2x}$

26/70 $y(x) = \dfrac{e^x}{1+2e^x}$. SOL: $y = \dfrac{e^x}{1+2e^x} \Rightarrow y + 2ye^x = e^x$

$\Rightarrow y = e^x(1-2y) \Rightarrow \dfrac{y}{1-2y} = e^x \Rightarrow x = \ln\left(\dfrac{y}{1-2y}\right) \Rightarrow$

$f^{-1}(x) = \ln\left(\dfrac{x}{1-2x}\right)$.

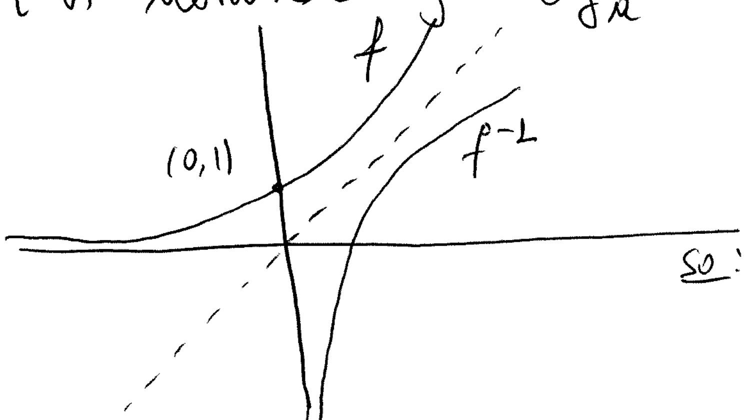
THE GRAPH of f^{-1} IS OBTAINED by REFLECTING the graph of f about the line $y = x$.



PARTICULAR CASE: $f(x) = a^x$; $a > 0$ — the exponential function.

FROM its graph $\Rightarrow$ it is 1-1. So it has an inverse.

it is denoted by $\log_a$: $f(x) = a^x = y \Longleftrightarrow \log_a y = x$



- $f: \mathbb{R} \to \mathbb{R}_+$
- $f^{-1} = \log_a : \mathbb{R}_+ \to \mathbb{R}$

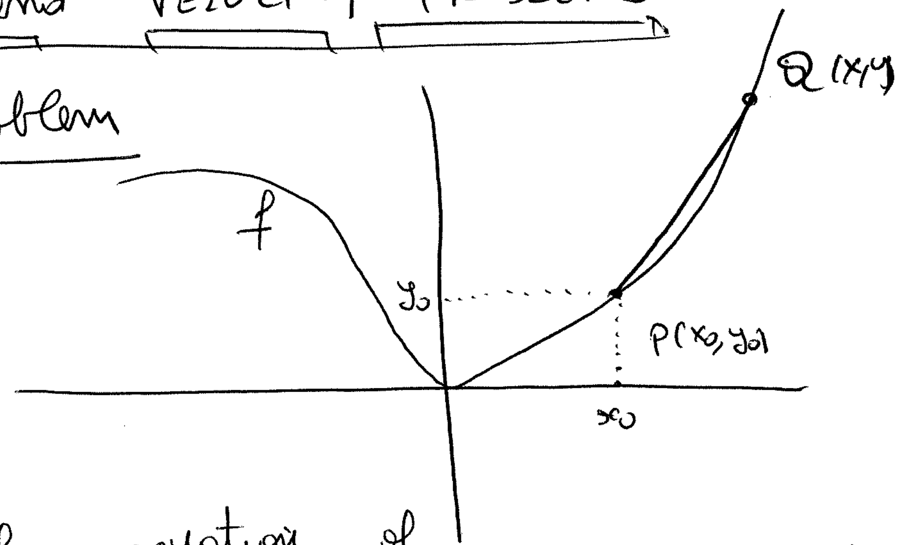
SO: • $\log_a a^x = x$; $a^{\log_a x} = x$

IMPORTANT IDENTITIES: $\sin^2\theta + \cos^2\theta = 1$, $\sin(x \pm y) = \dots$, $\cos(x \pm y) =$
$\sin 2x = 2\cos x \sin x$; $\cos 2x = \cos^2 x - \sin^2 x \dots$

CH. 2 LIMITS and DERIVATIVES

2.1 TANGENT and VELOCITY PROBLEMS

I The tangent problem



- tangent line = ~~tangent~~ line that touches the graph only in 1 point

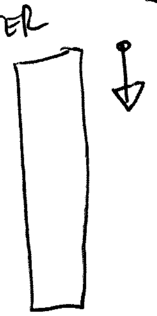
- PROBLEM: Find the equation of the tangent line. SO we need the SLOPE of the tangent line

- IDEA:
  - Pick Q (x,y) on the graph of f
  - join Q and P, compute the slope of PQ: $\frac{y-y_0}{x-x_0}$
  - when Q approaches P, the slope of the tangent line is the limit of the slopes of the secant lines QP:

$$m = \lim_{\substack{Q \to P \\ Q(x,y)}} \frac{y-y_0}{x-x_0}$$

II The velocity problem

TOWER



- the distance after t seconds

  $s(t) = (4.9)\,t^2$

- Q: What is the velocity after 5sec?

(Def) We write $\lim_{x\to a} f(x)=L$ and say "the limit of $f(x)$, as x approaches a, is L" if we can make the values of $f(x)$ arbitrary close to L, by taking x sufficiently close to a (on either side) but not equal to a

EXP: $\lim_{x\to 1} \frac{x-1}{x^2-1} = \frac{0}{0} = \lim_{x\to 1} \frac{(x-1)}{(x-1)(x+1)} = \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{1+1} = \frac{1}{2}$

DO: 21/103 $\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$

SOL: it is $\frac{0}{0}$; $= \lim_{x\to 0} (\sqrt{x+4}-2) \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} \cdot \frac{1}{x} =$

$= \lim_{x\to 0} \frac{x+4-4}{\sqrt{x+4}+2} \cdot \frac{1}{x} = \lim_{x\to 0} \frac{1}{\sqrt{x+4}+2} = \frac{1}{4}$

SIMILARLY one may define:

$\lim_{x\to a^-} f(x)=L$ and $\lim_{x\to a^+} f(x)=L$

? 7/102 $f(x)=\begin{cases} 1+x; & x<-1 \\ x^2 & ; -1\le x<1 \\ 2-x; & x\geq 1 \end{cases}$

SOL: $\begin{cases} \lim_{x\to -1^-} f(x)= \lim_{x\to -1^-} 1+x = 1+(-1)=0 \\ \neq \\ \lim_{x\to -1^+} f(x)= \lim_{x\to -1^+} x^2 = (-1)^2=1 \end{cases}$

$\begin{cases} \lim_{x\to 1^-} f(x)= \lim_{x\to 1^-} x^2 = 1^2=1 \\ \lim_{x\to 1^+} f(x)= \lim_{x\to 1^+} 2-x = 2-1=1 \end{cases}$

MOREOVER: The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is $f'(a)$.

DO: 16 a, b / 143

24/143

18/143 Sol: $y = mx + b$; $m = g'(5) = 4 \Rightarrow y = 4x + b$. Since $(5, -3)$ is on the graph $\Rightarrow -3 = 4 \times 5 + b \Rightarrow b = -23$

20/143 SOL: $(4, 3) = (a, f(a)) \Rightarrow a = 4$; $f(a) = 3 \Rightarrow f(4) = 3$

$y = mx + b \Rightarrow 2 = m \cdot 0 + b \Rightarrow b = 2 \Rightarrow y = mx + 2$

$m = \frac{3-2}{4-0} = \boxed{\frac{1}{4}} \Rightarrow f'(4) = m = \frac{1}{4}$

26/143 a)

34, 36, 38 / 143

Lecture 3 : 26th of September

§2.6

DO: $\boxed{16\ a,b/143}$ FOR a)

Sol: $s(t) = t^2 - 8t + 18$

ii) $\dfrac{s(4)-s(3.5)}{4-3.5} = \dfrac{(16-32+\cancel{18}) - [(3.5)^2 - 8(3.5) + \cancel{18}]}{0.5}$

$= \dfrac{-16 - 12.25 + 28}{0.5} = \dfrac{-0.25}{0.5} = -0.5.$

The same with the rest.

FOR b) $\lim\limits_{x\to 4} \dfrac{f(x)-f(4)}{x-4} \overset{x=t}{=} \lim\limits_{x\to 4} \dfrac{t^2-8t+18-4^2+8\cdot 4-18}{t-4} =$

$= \lim\limits_{t\to 4} \dfrac{(t-4)(t+4) - 8(t-4)}{(t-4)} = \lim\limits_{t\to 4} \dfrac{(t-4)[t+4-8]}{(t-4)} =$

$\lim\limits_{t\to 4} t+4-8 = 4+4-8=0.$ Is (for you) close enough to the result in ii)?

DO: 24/143 - EASY -

DO: 18/143 EASY

DO: 20/143 $(4,3) = (a, f(a)) \Rightarrow a=4;\ f(a)=3 \Rightarrow f(4)=3$

2 points on the tangent line: $(4,3)$ and $(0,2) \Rightarrow m = \dfrac{3-2}{4-0} = \dfrac{1}{4}$

So $f'(4) = m = \dfrac{1}{4}.$

26/143 a) easy ....

DO: 34, 36, 38/143 easy

MORE @ DGD on MON!

DO: 32 / 157 | Use Def + Split....

OTHER NOTATIONS: if $y = f(x)$ another notation is $f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{df(x)}{dx} = Df(x)$

DEF: A function is called differentiable at a **iF** $f'(a)$ exists. If f is differentiable at every number in an interval, then we call f differentiable on that interval

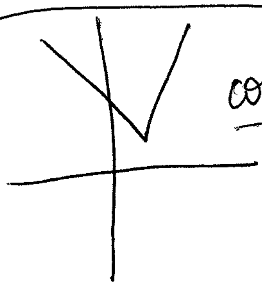
EXP: When is $f(x) = |x|$ differentiable?

SOL: On $\mathbb{R} \setminus \{0\}$. Do the computations....

THM If f is differentiable at a, then f is continuous at a.

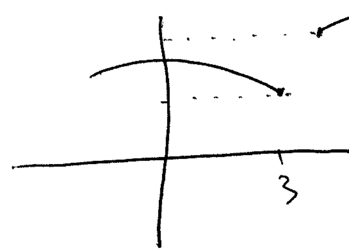
(what is a continuous fn? f is cont at a iF $\lim_{x \to a} f(x) = f(a)$.
Exactly what you expect: you may DRAW the graph without lifting the pencil....

WHEN is a fn. NOT DIFFERENTIABLE?



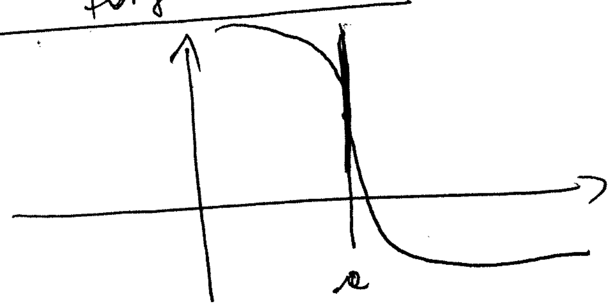
corners (i.e. no tangent in the corner).

jump



discontinuity (see the above thm)

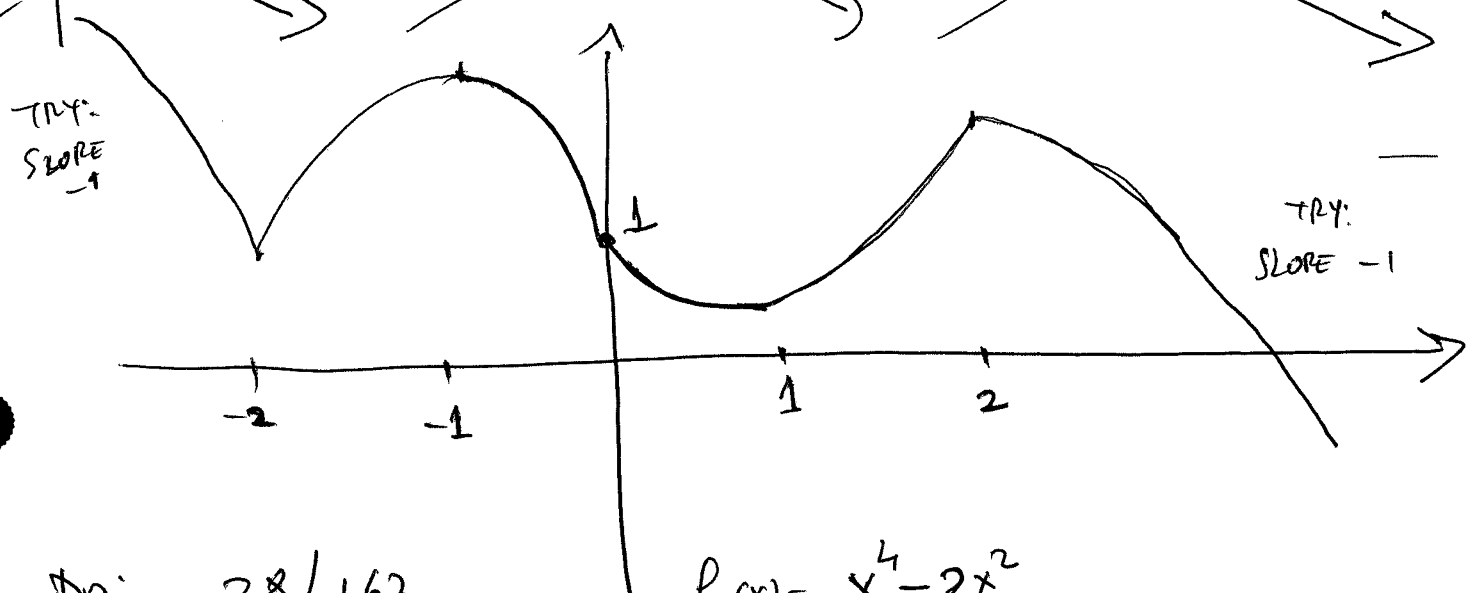
vertical tangent line:



DEF: An inflection point is a point where a curve changes its <u>concavity</u>.

INFL. POINT

x	$-\infty$	-2		-1	0	1		2	∞
f'	$-1-1-1-1$		$++++$	0	$----$	0	$+++++$		$-1-1-1-1$
f''			$-------$			$+++++$			
f	↘		↗ ∩		↘	∪	↗		↘



DO: 28/163 $\quad f(x) = x^4 - 2x^2$

SOL: a) $f'(x) = \ldots \text{etc} \ldots = 4x^3 - 4x$

$f''(x) = 12x^2 - 4 = 4(3x^2 - 1)$

b) $f'(x) > 0 \iff 4x(x^2-1) = 4x(x-1)(x+1)$

x	$-\infty$		-1		$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1		∞
f'	$----$		0	$++++$		0	$-$	0	$+++$	
f''		$+++++$			0	$---$	0		$+++$	
f	↘			↗	↗	↘	↘		↗	
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Lecture -4- OCT-3rd - 2009

DO: 18, 20, 24, 10 / 181 AND 28, 30, 32, 29 / 181

DO: 47, 54 / 182

SCAN all lectures before Test 1

3.2 PRODUCT and QUOTIENT rules

THM The PRODUCT RULE: If f, g are both differentiable then $(f(x)\cdot g(x))' = f'(x)g(x) + f(x)\cdot g'(x)$

THM The QUOTIENT rule: if f, g are both differentiable then $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}$.

DO: 36 a; 30 / 189; 24, 18, 34 a, 40, 20, 4 / 188

3.3 DERIVATIVES of TRIGONOMETRIC FUNCTIONS

BASED on $\begin{cases} \lim\limits_{x\to 0} \frac{\sin x}{x} = 1 & \text{(Try a calculator...)} \\ \text{and } \lim\limits_{x\to 0} \frac{\cos x - 1}{x} = 0 \end{cases}$

ONE GETS:

THM $\boxed{(\sin x)' = \cos x \quad \text{and} \quad (\cos x)' = -\sin x}$

TABLE / 194

$$\boxed{\begin{array}{ll} (\tan x)' = \sec^2 x \ ; & (\cot x)' = -\csc^2 x \\ (\sec x)' = (\sec x)(\tan x) \ ; & (\csc x)' = (-\csc x)\cot x \end{array}}$$

where $\sec x = \frac{1}{\ }$ and $\csc x = \frac{1}{\sin x}$

DO: 12/205, 32/205, 26/205, 44/205

46 a/205

50: SOL: $y'(x) = 2\cos(2x) - 2\cos x$

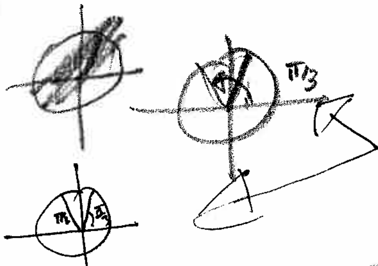
SOLVE: $2\cos(2x) = 2\cos x$; $\boxed{\cos 2x = \cos x, \quad 2\cos^2 x - 1 = \cos x}$

$\cos x = a \Rightarrow 2a^2 - a - 1 = 0 \qquad 2a^2 - 2a + a - 1 = 0$

$2a(a-1) + (a-1) = 0 \Rightarrow (a-1)(2a+1) = 0$

$$\begin{cases} a = 1 \\ a = -\frac{1}{2} \end{cases} \Rightarrow \begin{cases} \cos x = 1 \\ \text{or} \\ \cos x = -\frac{1}{2} \end{cases} \Rightarrow \begin{cases} x = 2k\pi, \ k \text{ in } \mathbb{Z} \\ x = \frac{2}{3}\pi + 2k\pi, \ k \text{ in } \mathbb{Z} \end{cases}$$

$\boxed{\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}}$



$\pi - \frac{\pi}{3} = \frac{2\pi}{3}$

- POWER rule + chain Rule: $\left((g(x))^n\right)' = n\, g(x)^{n-1} \cdot g'(x)$
- $(a^x)' = a^x \ln a$

52/205

3.5 Implicit Differentiation

When $y = f(x)$ is "given" one may find $y'(x)$.

What if y is given NOT! What if it is given: a relation?

SEE PLAN

10/214 $\qquad 1 + x = \sin(xy^2)$ WHO can find y?

we do: $0 + 1 = \cos(xy^2)\left[1 \cdot y^2 + x\, 2y \cdot y'\right]$

$$\frac{\frac{1}{\cos xy^2} - y^2}{2xy} = \boxed{y'(x)}$$