Comments on lecture 1

1.1 Some ways to represent functions

DEF: Let A, B be 2 sets (usually subsets of $\mathbf{R}$ - the set of real numbers). A function $f : A \mapsto B$ is a rule that assigns to each element $x \in A$ a unique element, $f(x)$, in B.

• A is called the domain of the function f; the set of all possible values of $f(x)$ is called the range of the function.

Example 0.0.1. *$f : \mathbf{R} \mapsto \mathbf{R}$, $f(x) = 2x + 1$ for all $x \in \mathbf{R}$. What does it represent geometrically?*

TO VISUALIZE:

DEF: *THE GRAPH* of a function $f : A \mapsto B$ is the set $\{(x, f(x)) | x \in A\}$

Example 0.0.2. *Graph $f(x) = 2x + 1$ for all $x \in \mathbf{R}$. Note that $(0, 1)$ and $(1, 3)$ are in the graph, then just join them BY a line!*

Example 0.0.3. *Graph $f(x) = 2x^2$ for all $x \in \mathbf{R}$. Note that $(0, 0)$, $(1, 2)$ and $(-1, 2)$ are in the graph, then just join them by curve! What is the shape?*

The ways to represent a function are:

a) — verbally (using words)

b) — numerically (table of values)

c) — algebraically (using formulae)

d) — visually (using a graph)

You already encountered c), d). For a) think about $P : \mathbf{R} \mapsto \mathbf{R}$, $P(t)$ is the human population of Ottawa at time t. For b) think about a cheap example as follows:

x	1	2	3
$f(x)$	2	-1	-1

Do: 30,32/page 23

FOR FUTURE LECTUREs (WE NEEDED FOR WHAT IS CALLED CALCULUS):

DEF: The difference quotient is given by $\frac{f(a+h)-f(a)}{h}$, where $h \neq 0$ (and of course f and a are given).

Do: 26,28/page 23

Natural Question: Is every curve (in the xy-plane) the graph of a function?

The answer is :

The Vertical Line Test: A curve in the xy-plane is the graph of a function (of x) if and only if ($\Leftrightarrow$) no vertical line cuts the curve more than once.

Example 0.0.4. *Think about the circle, parabola and other graphs...*

DEF: A PIECEWISE DEFINED FUNCTION is a function defined by different formulae in different parts of the domain.

Example 0.0.5. $f(x) = \begin{cases} x + \frac{7}{2} & \text{if } x \geq -1\,, \\ -2 & x < -1\,. \end{cases}$

Graph it!

Important Example (The absolute value)

If $a \in \mathbf{R}$, then $|a| = \begin{cases} a & \text{if } a \geq 0\,, \\ -a & \text{if } a < 0\,. \end{cases}$ is called the absolute value of a. Define now the function $f : \mathbf{R} \mapsto \mathbf{R}_+$, $f(x) = |x|$. What is the range? Graph it!

Do: 42/page 23 and READ example 9/page 19.

Symmetry DEF: a) A function f is called EVEN if $f(x) = f(-x)$ for all x in its domain.

b) A function is called ODD if $f(x) = -f(-x)$ for all x in its domain.

Example 0.0.6. $f(x) = x^2$, $f(x) = 2x^3$

Significance: a) The graph of an even function is symmetric with respect to the y-axis;

b) The graph of an odd function is symmetric about the origin $(0, 0)$.

Do: 72,70 /page 25

DECREASING AND INCREASING FUNCTIONS

It is what you have expected: **DEF:** $f : I \mapsto \mathbf{R}$ is called increasing on I if $f(x_1) < f(x_2)$ provided $x_1 < x_2$ in I.

DEF: $f : I \mapsto \mathbf{R}$ is called decreasing on I if $f(x_1) > f(x_2)$ provided $x_1 < x_2$ in I.

Question: Is $f(x) = x^2$ increasing?

1.2 Important functions

a) **Linear functions:** $f : \mathbf{R} \mapsto \mathbf{R}$, $f(x) = mx + b$, where m, b are real numbers. Slope is m; b is the y-intercept.

Do: 16/page 36

b) **Polynomials:** $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$; degree n if $a_n \neq 0$; $a_n, a_{n-1}, \ldots, a_1, a_0$ are called the coefficients of the polynomial.

Example 0.0.7. *1) A cubic function is just a polynomial of degree 3:* $P(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$, $a_3 \neq 0$

2) A quadratic is just a polynomial of degree 2: $P(x) = a_2 x^2 + a_1 x + a_0$, $a_2 \neq 0$. *Graph it for* $\pm$ *leading coefficient!*

c) **Power function:** $f(x) = x^a$, where a is a real number.

Important:

— If $a = -1$, $f(x) = x^{-1} = \frac{1}{x}$, $x \neq$

— If $a = \frac{1}{n}$, n positive integer, $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$

d) **Rational function:** $f(x) = \frac{P(x)}{Q(x)}$, where P, Q are polynomials.

e) **Algebraic function:** One obtained from polynomials using $+, -, \times, \div, \sqrt[n]{}$.

Example: $f(x) = \sqrt[7]{x - \frac{x}{2x+5}}$

f) **Trigonometric functions:** Sine, cosine, tangent = sin over cos etc...

g) **Exponential function:** $f(x) = a^x$, where a — the base — is a positive number; graph $f(x) = 2^x$, $g(x) = (\frac{1}{5})^x$.

h) **Logarithmic function:** $f(x) = \log_a x$, the base a is a positive number. It is the *inverse* of the EXP function.

NOTE THAT: $\log_a x = y$ iff $x = a^y$.

Do: 2/page 35

1.3 Getting new functions from old ones

Recall (otherwise read) the TRANSLATIONS (TO THE LEFT, RIGHT, UPWARD, DOWNWARD) + STRETCHING and REFLECTING

MORE IMPORTANT:

Combinations Given f and g one may define: $f+g$, $f-g$, fg, $\frac{f}{g}$ as follows

$(f+g)(x) = f(x)+g(x)$, $(f-g)(x) = f(x)-g(x)$, $(fg)(x) = f(x)g(x)$, $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$. The domain of the last one is the set of all x such that $g(x) \neq 0$ and $f(x)$ is defined. Find the domains of the others!

Do: 30/page 44

COMPOSITION Given f and g one defines their composition $f \circ g$ as follows $(f \circ g)(x) = f(g(x))$; what about the domain?

Do: 48,34,46/pages 44;45

1.5 EXPONENTIAL FUNCTIONS

DEF: $f(x) = a^x$, where $a > 0$

WHAT IS IT?

— If $x = n > 0$ then $a^x = a^n = a \times a \times \ldots a$

— If $x = 0$ then $a^0 = 1$

— If $x = -n < 0$ then $a^x = a^{-n} = \frac{1}{a^n} = (\frac{1}{a})^n$

— If $x = \frac{p}{q}$ is rational then $a^x = a^{\frac{p}{q}} = \sqrt[q]{a^p}$ etc

— If x is irrational...

Laws of exponents:

— $a^{x+y} = a^x a^y$

— $a^{x-y} = \frac{a^x}{a^y}$

— $(a^x)^y = a^{xy}$

— $(ab)^x = a^x b^x$

The Number e

e is the positive number a such that the slope of the tangent line (=?) at $(0,1)$ to $f(x) = a^x$ IS 1. For the moment $e \approx 2.71828$

Do: 20,30/pages 59

Comments on lecture 4

3.5 Some exercises

Exc 12. We have (by differentiating both sides) that $y'(\sin(x^2)) + y\cos(x^2)(2x) = 1\sin(y^2) + x\cos(y^2)(2y)(y')$, so $y'(\sin(x^2) - x\cos(y^2)(2y)) = \sin(y^2) - 2yx\cos(x^2)$, hence $y' = \frac{\sin(y^2)-2yx\cos(x^2)}{\sin(x^2)-x\cos(y^2)(2y)}$.

Exc 16. We have (by differentiating both sides, and by product Rule) that $\cos(x) - \sin(y)y' = \cos(x)\cos(y) + \sin(x)(-\sin(y))y'$, hence $y'\{-\sin(y)+\sin(x)\sin(y)\} = \cos(x)\cos(y) - \cos(x)$. Therefore $y' = \frac{\cos(x)\cos(y)-\cos(x)}{-\sin(y)+\sin(x)\sin(y)}$.

Exc 18. We have (by differentiating both sides, and by product Rule) that $g'(x) + 1\sin(g(x)) + x\cos(g(x))g'(x) = 2x$. Plug in $x = 0$ and get $g'(0) + \sin(g(0)) = 0$, therefore $g'(0) = -\sin(g(0))$.

Exc 22. Say the equation is $y = mx+n$, where $m = y'(\pi)$. But $\cos(x+y)\{1+y'\} = 2-2y'$, hence $y' = \frac{2-\cos(x+y)}{\cos(x+y)+2}$. From here we get $y'(\pi) = \frac{2-\cos(\pi+\pi)}{\cos(\pi+\pi)+2} = \frac{2-1}{1+2} = \frac{1}{3}$, so $m = \frac{1}{3}$, thus $y = \frac{1}{3}x + n$. From $\pi = (1/3)\pi + n$ we get $n = (2/3)\pi$, so you may write the equation!

Exc 24. Say the equation is $y = mx+n$, where $m = y'(1)$. But $2x+2y+2xy'-2yy'+1 = 0$, hence $y' = \frac{-1-2y-2x}{2x-2y}$, thus $m = \frac{7}{2}$. Since $2 = \frac{7}{2} \times 1 + n$, one gets $n = -3/2$.

Exc 28. Just doo it!

Comments on lecture 6

We did 3.7, 3.9, 4.1, and 4.2