Lecture : NOV-28-2009   1320E

Do: 47 ; 55/382   § SUB = 5.5

§ 5.6 INTEGRATION BY PARTS

PRODUCT RULE SAYS: $(f(x)\cdot g(x))' = f'(x)\cdot g(x) + f(x)\cdot g'(x)$

So: $\int (f'(x)\cdot g(x) + f(x)\cdot g'(x))\,dx = f(x)\cdot g(x)$

$$\boxed{\int f'(x)\cdot g(x)\,dx = f(x)\cdot g(x) - \int f(x)\cdot g'(x)\,dx.}$$

FOR INDEFINITE integrals.

FOR DEFINITE INTEGRALS :

$$\boxed{\int_a^b f'(x)\cdot g(x)\,dx = f(x)\cdot g(x)\Big|_a^b - \int_a^b f(x)\cdot g'(x)\,dx}$$

$\hookrightarrow$ $f(b)g(b) - f(a)\cdot g(a)$

DO: 4 ; 6 ; 10 ; 12 ; 18 ; 16 ; 24 /387

DO: ~~28/388~~ OR 30/388

! Read EXP 4/385 (OR DO IT)

§ 5.7 Additional Techniques of Integration

I TRIG INTEGRALS

we must know:

$$\begin{cases} \sin^2 x + \cos^2 x = 1 \\ \sin 2x = 2\sin x\cos x \\ \cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x \end{cases}$$

Read: EXP 1,2/38

DO 2/393 $\int_0^{\pi/2} \cos^5 x\,dx = \int_0^{\pi/2} \cos x(\cos^4 x)\,dx = \int_0^{\pi/2} \cos x(\cos^2 x)^2\,dx$

$= \int_0^{\pi/2} \cos x(1-\sin^2 x)^2\,dx = \int_0^1 (1-u^2)^2\,du = \int_0^1 (1-2u^2+u^4)\,du$

$u = \sin x$

$du = \cos x\,dx$

$= u - 2\frac{u^3}{3} + \frac{u^5}{5}\Big|_0^1 = 1 - \frac{2}{3} + \frac{1}{5} = \text{etc}$

IDEA:

$$\int (\cos x) \times (\text{fu of } \sin x)\, dx$$

or

$$\int (\sin x) \times (\text{fu of } \cos x)\, dx$$

(4)/393 $\int \sin^3(mx)\, dx = \int \sin(mx) \sin^2(mx)\, dx = \int \sin(mx) \cdot (1-\cos^2(mx))\, dx$

$= \int (1-u^2)\left(-\frac{du}{m}\right) = -\frac{1}{m}\int (1-u^2)\, du = -\frac{1}{m}\left(u - \frac{u^3}{3}\right) + C$

$C \#$

$\cos(mx) = u$

$du = -\sin(mx) \cdot m\, dx$

$= -\frac{1}{m}\left(\cos mx - \frac{1}{3}\cos^3 mx\right) + C$

(I) $m \neq 0 \Rightarrow \frac{du}{-m} = \sin(mx)\, dx$

(II) $m = 0 \quad \int 0\, dx = C \quad ; C\#$

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(6) $\int_0^{\pi/2} \sin^2 x \cos^2 x\, dx \overset{A}{=} \int_0^{\pi/2} \left(\frac{\sin 2x}{2}\right)^2 dx =$

$(\sin 2x = 2\sin x \cos x)$

$= \frac{1}{4}\int_0^{\pi/2} \sin^2 2x\, dx = \frac{1}{4}\int_0^{\pi/2} \frac{1-\cos 4x}{2}\, dx$

$\cos 2\alpha = 1 - 2\sin^2\alpha \ : \ \alpha = 2x$

$= \frac{1}{8}\int_0^{\pi/2} (1-\cos 4x)\, dx = \frac{1}{8}\left[x - \frac{\sin 4x}{4}\right]\Big|_0^{\pi/2}$

$= \frac{1}{8}\left(\frac{\pi}{2}\right) = \frac{\pi}{16}$

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more on BOD

§ 5.7 + App G
Integration of RATIONAL FUNCTIONS by PARTIAL FRACTION

Ⓘ ALGEBRA   Ⓘⓘ CALCULUS

Ⓘ **FIRST** • $f(x) = \dfrac{P(x)}{Q(x)}$ → rational function

if degree $P(x) \geq$ degree of $Q(x)$ Do the LONG DIVISION:

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

$S(x)$ easy to integrate : polynomial !

Exp:

$$
\begin{array}{l|l}
x^5 - x^4 + x^3 - x^2 + x + 1 & x^2 - 1 \\
\hline
x^5 \quad\; - x^3 & x^3 - x^2 + 2x - 2 \\
\hline
-x^4 + 2x^3 - x^2 + x + 1 & \\
-x^4 \quad\;\; + x^2 & \\
\hline
2x^3 - 2x^2 + x + 1 & \\
2x^3 \quad\; - 2x & \\
\hline
-2x^2 + 3x + 1 & \\
-2x^2 \quad\; + 2 & \\
\hline
\boxed{3x - 1} &
\end{array}
$$

$$\boxed{\frac{x^5 - x^4 + x^3 - x^2 + x + 1}{x^2 - 1} = x^3 - x^2 + 2x - 2 + \frac{3x - 1}{x^2 - 1}}$$

• NEXT FACTOR $Q(x)$.

There are 4 cases !

(I) $Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)$ ← ALL DISTINCT

Then $\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x_1+b_1} + \frac{A_2}{a_2x_2+b} + \cdots + \frac{A_n}{a_nx_n+b}$

(In fact $\frac{R(x)}{Q(x)} = \frac{R(x)}{(a_1x_1+b_1)\cdots(a_nx_n+b_n)} = \cdots$)

Q: HOW DO WE integrate $\frac{A_1}{a_1x_1+b_1}$?

A: $\int \frac{A_1}{a_1x+b_1}\,dx = A_1\int \frac{1}{a_1x+b_1}\,dx = A\int \frac{1}{u}\cdot\frac{1}{a_1}\,du$

$u = a_1x+b_1$
$du = a_1\,dx$

$= \frac{A}{a_1}\ln|u| + C = \frac{A}{a_1}\ln|a_1x+b_1| + C$

(II) EXC: DO: 10/ A54 Appendix

$$\int \frac{1}{(t+4)(t-1)}\,dt$$

SOL: $\frac{1}{(t+4)(t-1)} \overset{!}{=} \frac{A}{t+4} + \frac{B}{t-1} \Rightarrow$

$\frac{1}{(t+4)(t-1)} = \frac{At-A+Bt+4B}{(t+4)(t-1)}$

$1 = t(A+B) + (-A+4B)$ $\qquad \begin{cases} A+B=0 \\ -A+4B=1 \end{cases}$

$\begin{cases} A=-B \\ -A = 1-4B = 1+4A \Rightarrow A = \frac{1}{-5} \Rightarrow B = \frac{1}{5} \end{cases}$

$= \int \frac{-1/5}{t+4}\,dt + \int \frac{\frac{1}{5}}{t-1}\,dt = -\frac{1}{5}\ln|t+4| + \frac{1}{5}\ln|t-1| + C$

DO: 12/ A54 <u>very Impt!!</u>

- $$\int_0^1 \frac{x-1}{x^2+3x+2}\,dx$$

SOL: FACTOR $x^2+3x+2 = (x+1)(x+2)$

$$\frac{x-1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$x-1 = A(x+2) + B(x+1) \Rightarrow x-1 = x(A+B) + (2A+B)$$

$$\begin{cases} 1 = A+B \\ -1 = 2A+B \end{cases} \Rightarrow \begin{cases} A = 1-B \\ -1 = 2-2B+B \end{cases} \quad \begin{cases} A = 1-B \\ B = 3 \end{cases} \Rightarrow \begin{cases} A = -2 \\ B = 3 \end{cases}$$

$$\int_0^1 \frac{-2}{x+1}\,dx + \int \frac{3}{x+2}\,dx = -2\ln|x+1|\Big|_0^1 + 3\ln|x+2|\Big|_0^1$$

$= $ etc..... $\boxed{\text{DO}}$ the SUB !!

- II $Q(x) = (a_1x+b_1)^k \ldots\ldots$

Then $$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \cdots + \frac{A_k}{(a_1x+b_1)^k} + \cdots\cdots$$

Then <u>again</u>: DO A SUB .... easy; $u = (a_1x+b_1)^e \ldots$

DO 15/ A54

$$\int_3^4 \frac{x^3-2x^2-4}{x^3-2x^2}\,dx$$

SOL: LONG DIV.

$$\begin{array}{l|l} x^3-2x^2-4 & x^3-2x^2 \\ \underline{x^3-2x^2} & 1 \\ -4 & \end{array}$$

- $$\int_3^4 \left(1 + \frac{-4}{x^2(x-2)}\right)dx = x\Big|_3^4 - 4\int_3^4 \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}\right)dx$$

$$= 1 - 4\left(A \ln x\Big|_3^4 + B \cdot \frac{x^{-1}}{-1}\Big|_3^4 + C \ln|x-2|\Big|_3^4\right)$$

etc....

III $Q(x) = \underbrace{(ax^2+bx+c)}_{\text{(only one)}}$ $(\Delta<0)$ .... $\underbrace{\qquad\qquad}_{\text{other factors}}$

Then $\dfrac{R(x)}{Q(x)} = \dfrac{Ax+B}{ax^2+bx+c} + \ldots$

Q: How do we integrate $\displaystyle\int \frac{Ax+B}{ax^2+bx+c}\,dx$

A: Use $\boxed{\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C}$ C#

or complete the square, and then use....

DO: (22)/ A 54

$$\int \frac{x^2-x+6}{x^3+3x}\,dx$$

Sol: $Q(x) = x(x^2+3)$

$$\frac{x^2-x+6}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

$$x^2-x+6 = Ax^2+3A+Bx^2+Cx$$

$$\left.\begin{aligned} 1 &= A+B \\ -1 &= C \\ 6 &= 3A \end{aligned}\right. \Rightarrow \left\{\begin{aligned} A &= 2 \\ B &= 1-2=-1 \\ C &= -1 \end{aligned}\right.$$

$$\int\left(\frac{2}{x} + \frac{-x-1}{x^2+3}\right)dx = 2\ln|x| - \int \frac{x+1}{x^2+3}\,dx$$

$$= 2\ln|x| - \int \frac{x}{x^2+3}\,dx - \int \frac{1}{x^2+(\sqrt{3})^2}\,dx$$

$$= 2\ln|x| - \int \frac{du}{2\cdot u} - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

$u = x^2+3$

$du = 2x\,dx$

$$= 2\ln|x| - \frac{1}{2}\ln|x^2+3| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

IV $Q(x) = (ax^2+bx+c)^k \cdot \underbrace{\cdots\cdots\cdots}_{\text{other factors}}$

$(\Delta<0)$

<u>Then:</u> $$\frac{R(x)}{Q(x)} = \frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$$

$$+ \underbrace{\cdots\cdots}+\underbrace{\cdots\cdots}$$

<u>Do:</u> 31 / A54

$$\int \frac{1}{x(x^2+4)^2}\,dx$$

<u>Sol:</u> $$\frac{1}{x(x^2+4)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

$$1 = A(x^2+4)^2 + (Bx+C)x(x^2+4) + x(Dx+E)$$

$$1 = Ax^4 + 8x^2A + 16A + Bx^4 + B\cdot x^2\cdot 4 + Cx^3 + Cx4 + Dx^2 + Ex$$

$$\begin{cases} A+B=0 \\ C=0 \\ 8A+4B+D=0 \\ 4C+E=0 \\ 16A=1 \end{cases} \qquad \begin{cases} A=\frac{1}{16} \\ B=-\frac{1}{16} \\ C=0 \\ D=-4\cdot\frac{1}{-16}-8\cdot\frac{1}{16}=\frac{4}{-16}=-\frac{1}{4} \\ E=0 \end{cases}$$

so: $I = \int \frac{\frac{1}{16}}{x}\,dx + \int \frac{-\frac{1}{16}x}{x^2+4}\,dx +$

$+ \int \frac{-\frac{1}{4}\ x}{(x^2+4)^2}\,dx$

$= \frac{1}{16}\ln|x| - \frac{1}{16}\int \frac{x}{x^2+4}\,dx - \frac{1}{4}\int \frac{x}{(x^2+4)^2}\,dx$

$u = x^2+4$ etc $\quad u = x^2+4$ etc

TRY 14; 26 (use or NOT use partial fractions)

18, 19 / A54

more on pg #26/394