MAT 1320 E Fall 2009. November 14th Prof. Catalin Rada

TEST #2

Max = 20

Student Number: SOL

• Time: 80 min.

• Only basic scientific calculators are permitted: non-programmable, non-graphing, no differentiation or integration capability. Notes or books are not permitted.

• Work all problems in the space provided. Use the backs of the pages for rough work if necessary. **Do not use any other paper!**

• Write *only* in non-erasable ink (ball-point or pen), not in pencil. Cross out, if necessary, but do not erase or overwrite. Graphs and sketches may be drawn in pencil.

• Problems require complete and clearly presented solutions and carry part marks if there is substantial correct work toward the solution.

1. [2 points] If $8xy^{\frac{9}{2}} = 9 - x^{-\frac{7}{5}}$ find the slope of the tangent line to the curve y at the point $(1, 1)$.

$$8y^{9/2} + 8x \cdot \frac{9}{2} y^{7/2} \cdot y' = \frac{7}{5} x^{-12/5} \quad (1p)$$

so: $y'(1) = \frac{-11}{60}$ (by pluging in $(1,1)$) (1p)

2. [4 points] Use logarithmic differentiation to find the derivative of $f(x) = \dfrac{x^{2008} \tan x}{(x^2 + 2009)^{2009}}$.

$$\left.\begin{aligned} \ln f(x) &= \ln x^{2008} + \ln(\tan x) - \ln(x^2+2009)^{2009} \\ \ln f(x) &= 2008 \ln x + \ln(\tan x) - 2009 \ln(x^2+2009) \end{aligned}\right) \quad (2p)$$

$$\frac{f'(x)}{f(x)} = 2008 \cdot \frac{1}{x} + \frac{(\tan x)'}{\tan x} - 2009 \cdot \frac{2x}{x^2+2009} \quad (1p)$$

$$f'(x) = \frac{x^{2008} \cdot \tan x}{(x^2+2009)^{2009}} \cdot \left\{ \frac{2008}{x} + \frac{\frac{1}{\cos^2 x}}{\tan x} - \frac{4018x}{x^2+2009} \right\} \quad (1p)$$

3. [4 points] Find the following limits:

(a) $\lim_{x \to \pi} \dfrac{1+\cos(x)}{(x-\sin(x))^2}$

(b) Find f if $f''(x) = 6x + \sin(x)$.

b) $f'(x) = 6 \cdot \dfrac{x^2}{2} + (-\cos x) + C \;;\; C\#$ 1p

$f(x) = 3 \cdot \dfrac{x^3}{3} - \sin x + Cx + K, \quad \cancel{K\#}$ 1p

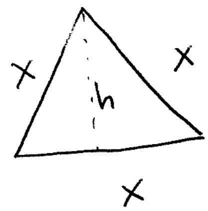
$f(x) = x^3 - \sin x + Cx + K \quad ; C, K \#$

a) $\lim_{x \to \pi} \dfrac{1+\cos x}{(x - \sin x)^2} = \dfrac{1+\cos(\pi)}{(\pi - \sin\pi)^2} = \dfrac{1+(-1)}{(\pi)^2} = 0$

2p

No: L'Hospital Rule....

4. [4 points] The side length of an equilateral triangle is increasing at a rate of 33 cm/min. How fast is the area increasing at the moment when area is 120 cm^2?



SIDE x ; $\frac{dx}{dt} = 33$

1p

$$A = \frac{x^2\sqrt{3}}{4}$$

since $h = \sqrt{x^2 - \frac{x^2}{4}} = \frac{x\sqrt{3}}{2}$

since $A = \frac{B \cdot h}{2} = \frac{x}{2} \cdot \frac{x\sqrt{3}}{2}$

$$\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt} = \frac{2x\sqrt{3}}{4} \cdot 33$$ by Chain Rule...

1p

From $\frac{x^2\sqrt{3}}{4} = 120 \Rightarrow x^2 = \frac{480}{\sqrt{3}} \Rightarrow x = \sqrt{\frac{480}{\sqrt{3}}}$

1p

So: $\frac{dA}{dt} = \frac{2\sqrt{3}}{4} \cdot \frac{\sqrt{480}}{\sqrt{\sqrt{3}}} \cdot 33 = \frac{2\cdot\sqrt{3}\cdot 4\sqrt{10}\cdot 33}{4\cdot\sqrt{\sqrt{3}}}$

1p

$$= \frac{2\sqrt{30}\cdot 33}{\sqrt{\sqrt{3}}} = \frac{66\sqrt{30}}{\sqrt{\sqrt{3}}}$$

5. [6 points] Consider the function $y = f(x) = \dfrac{x^3}{x^2-4}$.

(i) Find the intervals of increase and decrease and any local extrema.
(ii) Find the intervals of concavity and any inflection points.
(iii) Use all of the appropriate information to sketch the graph.

(i) $f'(x) = \dfrac{3x^2(x^2-4) - x^3(2x)}{(x^2-4)^2} = \dfrac{x^2(x^2-12)}{(x^2-4)^2}$ (1p)

x		$-\sqrt{12}$		0		$\sqrt{12}$	
f'	$+$	0	$-$	0	$-$	0	$+$ $+$
f	$\nearrow$		$\searrow$		$\searrow$		$\nearrow$

f is increasing on $(-\infty, -\sqrt{12}) \cup (\sqrt{12}, \infty)$
f is decreasing on $(-\sqrt{12}, \sqrt{12})$
$-\sqrt{12}$ is a loc MAX ; $\sqrt{12}$ is a local MIN
(1p)

(ii) $f''(x) = \dfrac{(4x^3 - 24x)(x^2-4) - 4x^3(x^3-12)}{(x^2-4)^3}$

$= \dfrac{8x^3 + 96x}{(x^2-4)^3} = 8x\,\dfrac{x^2+12}{(x^2-4)^3}$ (1p)

x		-2		0		2	
f''	$-$ $-$		$+$	0	$-$		$+$
f	$\cap$		$\cup$		$\cap$		$\cup$

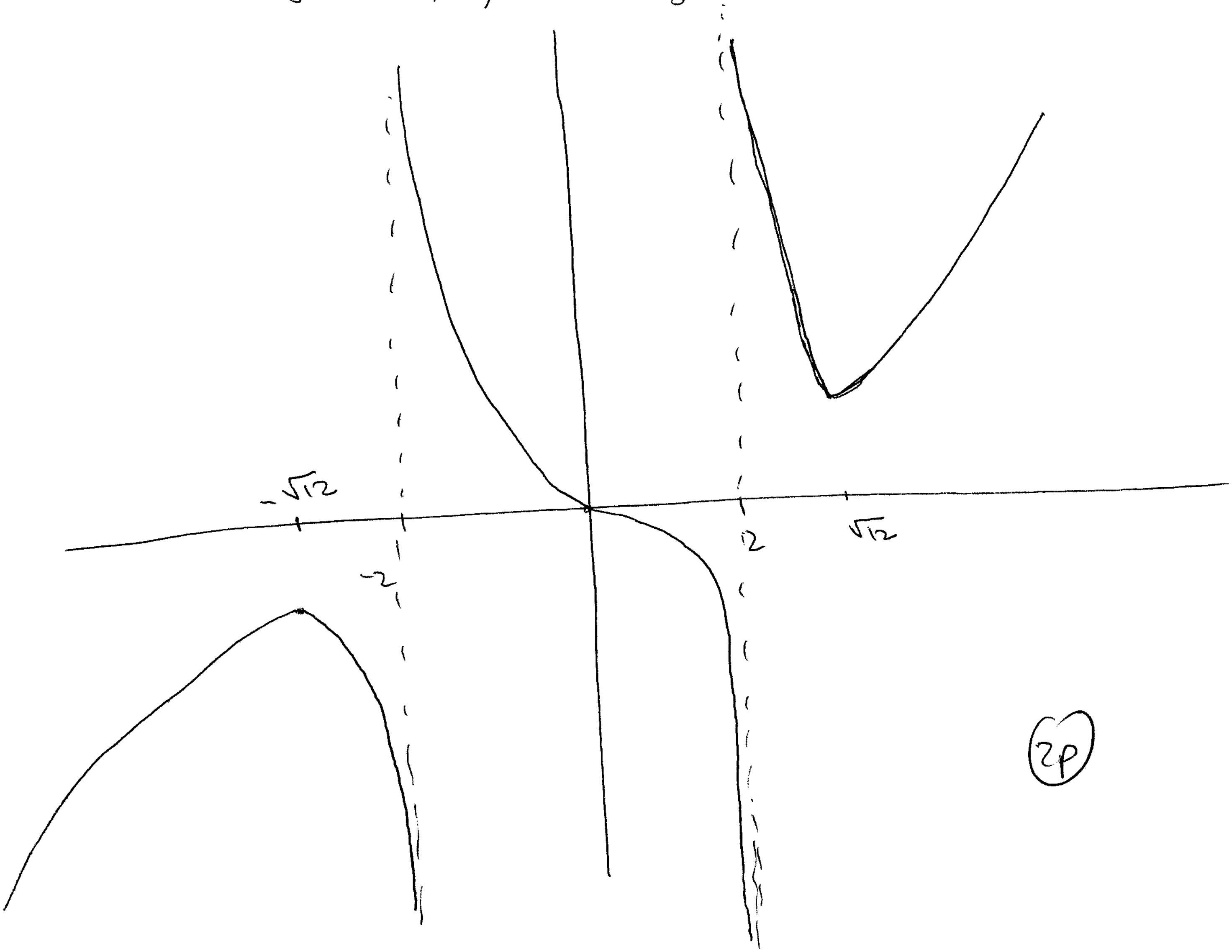
(1p)

f is concave up on $(-2, 0) \cup (2, \infty)$
f is concave Down on $(-\infty, -2) \cup (0, 2)$

$(0, 0)$ is the only inflection point

More space for question 5.

Looking at i, ii) one gets:



(2p)

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- Problems require complete and clearly presented solutions and carry part marks if there is substantial correct work toward the solution.

1. [2 points] If $xy^4 + x = 17$ find the equation of the tangent line to the curve y at each point where $x = 1$.

SOLVE (after $x=1$) $y^4 = 16 \Rightarrow (y-2)(y+2)(y^2+4)=0 \Rightarrow$ ($\frac{1}{2}$p)

$(1,2)$; $(+1,-2)$

NOW: $y^4 + x \cdot 4 \cdot y^3 \cdot y' + 1 = 0 \Rightarrow y' = \frac{-1-y^4}{4xy^3}$ ($\frac{1}{2}$p)

AT $(1,2)$: $y = mx+b = \frac{-17}{32}x + b \Rightarrow y = \frac{-17}{32}x + \frac{81}{32}$ since $(1,2)$ is on the tangent line. ($\frac{1}{2}$p)

AT $(1,-2)$ $y = mx+b = \frac{17}{32}x + b \Rightarrow y = \frac{17}{32}x - \frac{81}{32}$ since $(1,-2)$ is on the tangent line (0.5p) = ($\frac{1}{2}$p)

2. [4 points] Use logarithmic differentiation to find the derivative of $f(x) = \dfrac{x^{2009}\cos^{-1}x}{(x^2+1)^{2009}}$.

$\ln f(x) = \ln x^{2009} + \ln(\cos^{-1}x) - \ln(x^2+1)^{2009}$

$\ln f(x) = 2009 \ln x + \ln(\cos^{-1}x) - 2009 \cdot \ln(x^2+1)$ (2p)

$\frac{f'(x)}{f(x)} = 2009 \cdot \frac{1}{x} + \frac{(\cos^{-1}x)'}{\cos^{-1}x} - 2009 \cdot \frac{2x}{x^2+1}$ (1p)

SO:

$$f'(x) = \frac{x^{2009} \cdot \cos^{-1}(x)}{(x^2+1)^{2009}} \cdot \left\{ \frac{2009}{x} + \frac{\frac{-1}{\sqrt{1-x^2}}}{\cos^{-1}(x)} - \frac{4018x}{x^2+1} \right\}$$

(1p)

3. [4 points]

(a) Find the following limit: $\lim_{x \to 0} \frac{1}{\sin(x)} - \frac{1}{x}$

(b) Find f if $f''(x) = 2 + x^3 + x^6$.

1p

a) $$\lim_{x \to 0} \frac{x - \sin x}{x \cdot \sin x} = \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} =$$

$$= \lim_{x \to 0} \frac{\sin x}{\cos x + \cos x + x(-\sin x)} = 0$$

1p

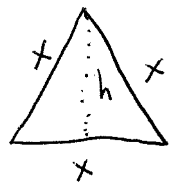
b) $$f'(x) = 2x + \frac{x^4}{4} + \frac{x^7}{7} + C; \quad C \#$$

1p

$$f(x) = x^2 + \frac{x^5}{20} + \frac{x^8}{58} + Cx + K; \quad C, K \#$$

1p

4. [4 points] The side length of an equilateral triangle is increasing at a rate of 3 cm/min. How fast is the area increasing at the moment when area is 10 cm^2?



SIDE x

$$\frac{dx}{dt} = 3$$

$$A = \frac{x^2\sqrt{3}}{4} \quad \left\{ \begin{array}{l} \text{since } h = \sqrt{x^2 - \frac{x^2}{4}} = \frac{x\sqrt{3}}{2} \\ \text{since } A = \frac{B \cdot h}{2} = \frac{x}{2} \cdot \frac{x \cdot \sqrt{3}}{2} \end{array} \right.$$

1p

$$\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt} = \frac{2x\sqrt{3}}{4} \cdot 3$$ by Chain Rule;

1p

From: $\frac{x^2 \cdot \sqrt{3}}{4} = 10 \Rightarrow x^2 = \frac{40}{\sqrt{3}}$; $x = \sqrt{\frac{40}{\sqrt{3}}} = \frac{2\sqrt{10}}{\sqrt{\sqrt{3}}}$ } 1p

So: $\frac{dA}{dt} = \frac{2\sqrt{3}}{4} \cdot 3 \cdot \frac{2 \cdot \sqrt{10}}{\sqrt{\sqrt{3}}} = \frac{3\sqrt{3} \cdot \sqrt{10}}{\sqrt{\sqrt{3}}} = \frac{3\sqrt{10}}{\sqrt{\sqrt{3}}}$ } 1p

5. [6 points] Consider the function $y = f(x) = \dfrac{5x}{(x-2)^2}$.

(i) Find the intervals of increase and decrease and any local extrema.
(ii) Find the intervals of concavity and any inflection points.
(iii) Use all of the appropriate information to sketch the graph.

(i) $$f'(x) = \frac{5(x-2)^2 - 5x(x-2)\cdot 2}{(x-2)^4} = \frac{-5(x+2)}{(x-2)^3}$$ (1p)

x		-2		2	
f'	$-$	0	$+$	‖	$-$
f	↘		↗	‖	↘

(1p)

f is increasing on $(-2, 2)$
f is Decreasing on $(-\infty, -2) \cup (2, \infty)$

$\boxed{-2 \text{ is a LOCAL min}}$

(ii) $$f''(x) = \frac{-5(x-2)^3 + 5(x+2)(x-2)^2 \cdot 3}{(x-2)^6} = \frac{10(x+4)}{(x-2)^4}$$ (1p)

x		-4		2	
$f''(x)$	$- - - -$	0	$+ + +$	‖	$+ + +$
f	⌢		⌣	‖	⌣

f is concave Down on $(-\infty, -4)$
f is concave up on $(-4, 2) \cup (2, \infty)$

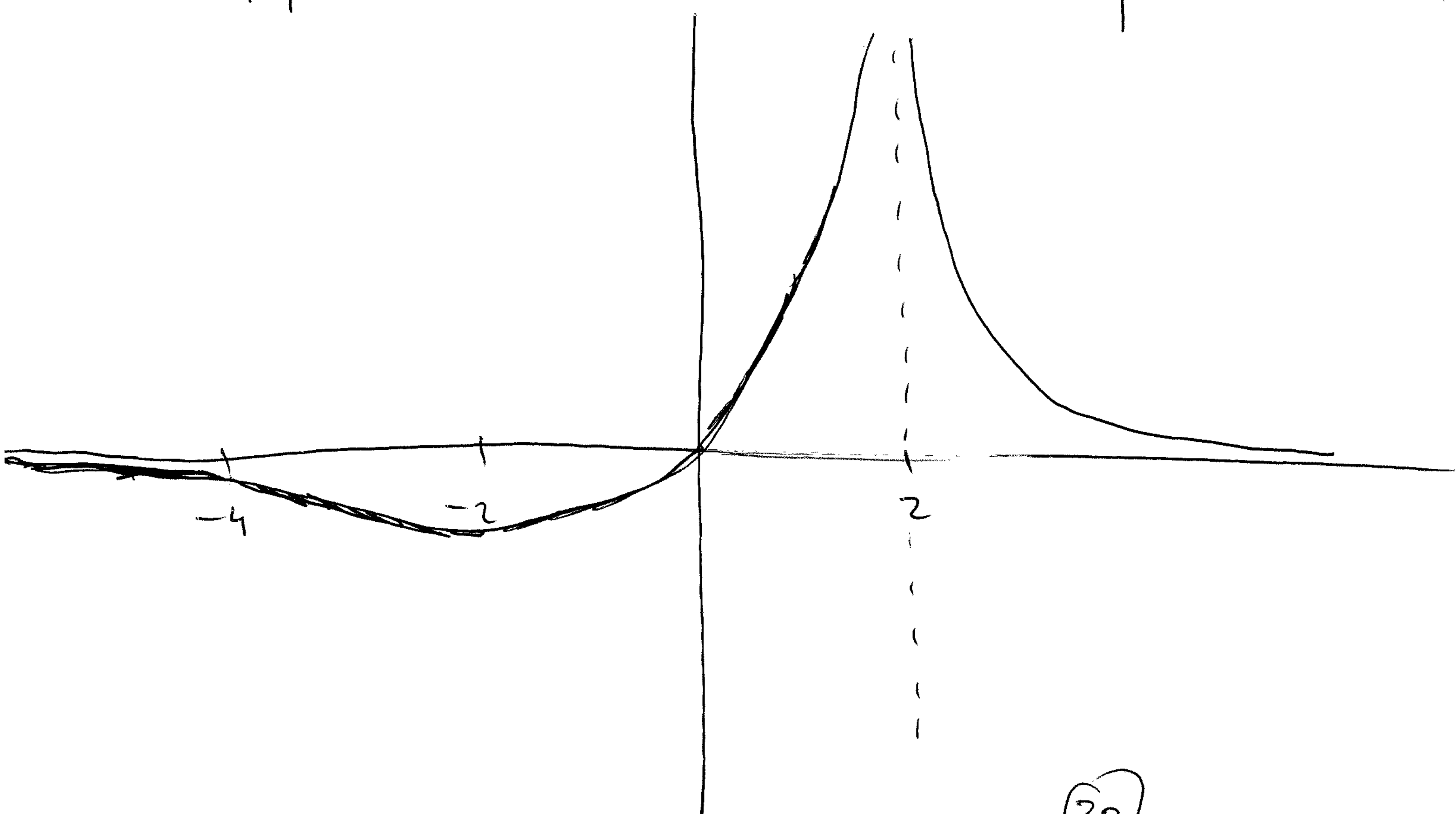
$\boxed{-4 \text{ is an inflection Point}}$

(1p)

More space for question 5.

MIN · LOC

x	$-\infty$		-4		-2		2		∞
f'		$- - -$	$-$	$-$	0	$+ + + +$		$- - - -$	
f''		$- - - - -$	0	$+ + +$			$+$	$+ + +$	
f	0						∞ | ∞		0



2p