University of Ottawa

Department of Mathematics and Statistics

MAT 1302A: Mathematical Methods II

Instructor: Catalin Rada

First Midterm Exam – February 6, 2008

Surname ______________________ First Name ______________________

Student # ______________________ DGD (1–4) ______________

Instructions:

(1) You have 80 minutes to complete this exam.
(2) The number of points available for each question is indicated in square brackets.
(3) You must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.
(4) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(5) Write your student number at the top of each page in the space provided.
(6) No notes, books, calculators or scrap paper are allowed.
(7) You should write in **pen**, not pencil.
(8) The final page of the exam may be used for scrap work.
(9) Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	6	4	5	5	5	5	30
Score							

1. [**6**] Is the following linear system consistent? If so, find the general solution.

$$\begin{aligned} v + 12 &= 3w + 3x - y \\ v + 2w + 5x + 2 &= w + 3y + 6 \\ -v + w - x + y - 4 &= 0 \\ -3w + y &= -12 - v + 3x \end{aligned}$$

Solution: We first write the system in standard form:

$$\begin{array}{rcrcrcrcr} v & - & 3w & - & 3x & + & y & = & -12 \\ v & + & w & + & 5x & - & 3y & = & 4 \\ -v & + & w & - & x & + & y & = & 4 \\ v & - & 3w & - & 3x & + & y & = & -12 \end{array}$$

We then write the augmented matrix and row reduce:

$$\left[\begin{array}{cccc|c} 1 & -3 & -3 & 1 & -12 \\ 1 & 1 & 5 & -3 & 4 \\ -1 & 1 & -1 & 1 & 4 \\ 1 & -3 & -3 & 1 & -12 \end{array}\right] \xrightarrow[R_4 - R_1 \to R_4]{\substack{R_2 - R_1 \to R_2 \\ R_3 + R_1 \to R_3}} \left[\begin{array}{cccc|c} 1 & -3 & -3 & 1 & -12 \\ 0 & 4 & 8 & -4 & 16 \\ 0 & -2 & -4 & 2 & -8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 - \frac{1}{2}R_2 \to R_3} \left[\begin{array}{cccc|c} 1 & -3 & -3 & 1 & -12 \\ 0 & 4 & 8 & -4 & 16 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{4}R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & -3 & -3 & 1 & -12 \\ 0 & 1 & 2 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 + 3R_2 \to R_1} \left[\begin{array}{cccc|c} 1 & 0 & 3 & -2 & 0 \\ 0 & 1 & 2 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since the rightmost column is not a pivot column, the system is consistent. We return to equation notation.

$$\begin{array}{rcrcrcr} v & & & + & 3x & - & 2y & = & 0 \\ & & w & + & 2x & - & y & = & 4 \end{array}$$

The basic variables are v and w and the free variables are x and y. Solving for the basic variables in terms of the free variables, we obtain the general solution to the system:

$$\begin{aligned} v &= -3x + 2y \\ w &= -2x + y + 4 \\ x, y &\quad \text{free} \end{aligned}$$

2. [**4**] Find all solutions to the vector equation

$$x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 14 \\ -4 \\ -14 \end{bmatrix}.$$

Solution: We reduce the corresponding augmented matrix to echelon form.

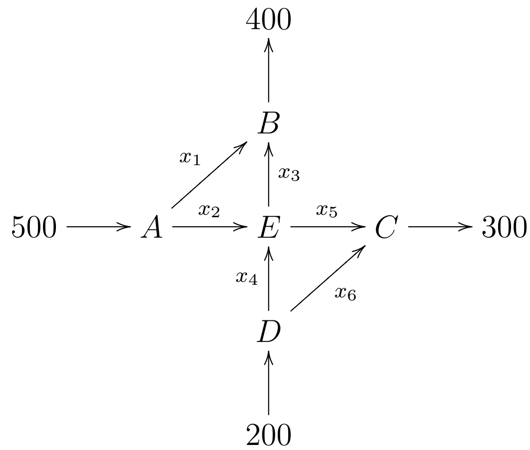
$$\left[\begin{array}{ccc|c} 1 & -3 & 3 & 14 \\ 0 & 2 & -2 & -4 \\ -1 & 5 & -1 & -14 \end{array}\right] \xrightarrow{R_3+R_1\to R_3} \left[\begin{array}{ccc|c} 1 & -3 & 3 & 14 \\ 0 & 2 & -2 & -4 \\ 0 & 2 & 2 & 0 \end{array}\right] \xrightarrow{R_3-R_2\to R_3} \left[\begin{array}{ccc|c} 1 & -3 & 3 & 14 \\ 0 & 2 & -2 & -4 \\ 0 & 0 & 4 & 4 \end{array}\right]$$

$$\xrightarrow[\frac{1}{4}R_3\to R_3]{\frac{1}{2}R_2\to R_2} \left[\begin{array}{ccc|c} 1 & -3 & 3 & 14 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_2+R_3\to R_2]{R_1-3R_3\to R_1} \left[\begin{array}{ccc|c} 1 & -3 & 0 & 11 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1+3R_2\to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Therefore the unique solution to the vector equation is

$$x_1 = 8, \quad x_2 = -1, \quad x_3 = 1.$$

3. Consider the traffic flow described by the following diagram. The letters A through E label intersections and numerical values indicate flow in cars per minute. The arrows indicate the direction of flow (all roads are one-way).



(a) [**2**] Write down a linear system describing the traffic flow, as well as all constraints on the variables x_i, $i = 1, \ldots, 6$. (Do not perform any calculations at this stage.)

Solution: Since the variables x_i denote numbers of cars per minute along one-way streets, we have

$$x_i \in \mathbb{Z} \quad \text{et} \quad x_i \geq 0 \quad \text{pour} \quad i = 1, ..., 6.$$

Setting the flow in equal to the flow out at each intersection, we get the following linear system.

Intersection	**Flow in**		**Flow out**
A	500	$=$	$x_1 + x_2$
B	$x_1 + x_3$	$=$	400
C	$x_5 + x_6$	$=$	300
D	200	$=$	$x_4 + x_6$
E	$x_2 + x_4$	$=$	$x_3 + x_5$

(b) [**2**] The reduced echelon form of the augmented matrix corresponding to the linear system in part (a) is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 400 \\ 0 & 1 & -1 & 0 & 0 & 0 & 100 \\ 0 & 0 & 0 & 1 & 0 & 1 & 200 \\ 0 & 0 & 0 & 0 & 1 & 1 & 300 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Find the general solution of the linear system. (Ignore the constraints for now.)

Solution: The basic variables are x_1, x_2, x_4 et x_5; and the free variables are x_3 et x_6. Writing the basic variables in terms of the free variables, we obtain the general solution to the system:

$$\begin{aligned} x_1 &= -x_3 + 400 \\ x_2 &= x_3 + 100 \\ x_4 &= -x_6 + 200 \\ x_5 &= -x_6 + 300 \\ x_3, x_6 &\quad \text{free} \end{aligned}$$

(c) [**1**] If the flow along EB is limited to a maximum of 100 cars per minute due to road work, what is the smallest possible flow along AB?

Solution: The constraints $x_i \geq 0$, for each $i = 1, \ldots, 6$ imply

$$\begin{aligned} x_1 \geq 0 &\implies x_3 \leq 400 \\ x_2 \geq 0 &\implies x_3 \geq -100 \\ x_3 \geq 0 &\implies x_3 \geq 0 \\ x_4 \geq 0 &\implies x_6 \leq 200 \\ x_5 \geq 0 &\implies x_6 \leq 300 \\ x_6 \geq 0 &\implies x_6 \geq 0 \end{aligned}$$

Thus $0 \leq x_3 \leq 400$ and $0 \leq x_6 \leq 200$. In addition, if $x_3 \leq 100$, the minimum flow along AB is

$$x_1 = -100 + 400 = 300.$$

4. [**5**] Let

$$\mathbf{a}_1 = \begin{bmatrix}1\\2\\0\end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix}1\\1\\-1\end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix}-1\\1\\3\end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix}4\\4\\1\end{bmatrix}.$$

Is the vector $\mathbf{b}$ in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$?

Solution: We row reduce the matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 2 & 1 & 1 & 4 \\ 0 & -1 & 3 & 1 \end{array}\right] \xrightarrow{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & -1 & 3 & -4 \\ 0 & -1 & 3 & 1 \end{array}\right] \xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & -1 & 3 & -4 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

Since the rightmost column is a pivot column, the corresponding system is inconsistent. Therefore, the answer is no, $\mathbf{b}$ is not in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

5. Let

(a) [**4**] Describe the general solution to the equation

$$\begin{bmatrix} 2 & -2 & 6 \\ 3 & -3 & 9 \\ -1 & 1 & -3 \end{bmatrix} \vec{x} = \begin{bmatrix} 8 \\ 12 \\ -4 \end{bmatrix}$$

in *vector parametric form.*

Solution: We row reduce the corresponding augmented matrix.

$$\left[\begin{array}{ccc|c} 2 & -2 & 6 & 8 \\ 3 & -3 & 9 & 12 \\ -1 & 1 & -3 & -4 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 4 \\ 3 & -3 & 9 & 12 \\ -1 & 1 & -3 & -4 \end{array}\right] \xrightarrow[R_3+R_1\to R_3]{R_2-3R_1\to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution is

$$\begin{aligned} x_1 &= x_2 - 3x_3 + 4 \\ x_2, x_3 &\quad \text{free} \end{aligned}$$

Thus, in vector parametric notation, the solution set is

$$\left\{ \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \middle| \, s, t \in \mathbb{R} \right\}$$

(b) [**1**] What is the solution set to the homogeneous equation

$$\begin{bmatrix} 2 & -2 & 6 \\ 3 & -3 & 9 \\ -1 & 1 & -3 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad ?$$

Describe it as the span of a collection of vectors. (Note that the coefficient matrix is the same as in part (a).)

Solution: The solution set to the homogeneous equation is

$$\text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

6. Consider the following linear system:

$$\begin{aligned} y + z &= -2 \\ x + 2y + z &= 1 \\ x - y + cz &= 7 \end{aligned}$$

(a) [**1**] Write down the augmented matrix $[\ A \mid \vec{b}\ ]$ of the system.
(b) [**2**] Find the reduced echelon form of $[\ A \mid \vec{b}\ ]$ when $c = -2$.
(c) [**2**] Find all possible reduced echelon forms of $[\ A \mid \vec{b}\ ]$ when c is arbitrary. Are there values of c for which the system is inconsistent?

Solution: The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & -2 \\ 1 & 2 & 1 & 1 \\ 1 & -1 & c & 7 \end{array}\right]$$

We row reduce this matrix. If one applies the four row operations (in order) $R_1 \leftrightarrow R_2$, $R_3 - R_1 \to R_3$, $R_3 + 3R_2 \to R_3$ and $R_1 - 2R_2 \to R_1$ one obtains

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & c+2 & 0 \end{array}\right]$$

If $c = -2$, the reduced echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

For all other values of c, the reduced echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

which is obtained by performing the following row operations: $R_3/(c+2) \to R_3$, $R_2 - R_3 \to R_2$ et $R_1 + R_3 \to R_1$. Since there is never a pivot in the rightmost column, the system is always consistent.